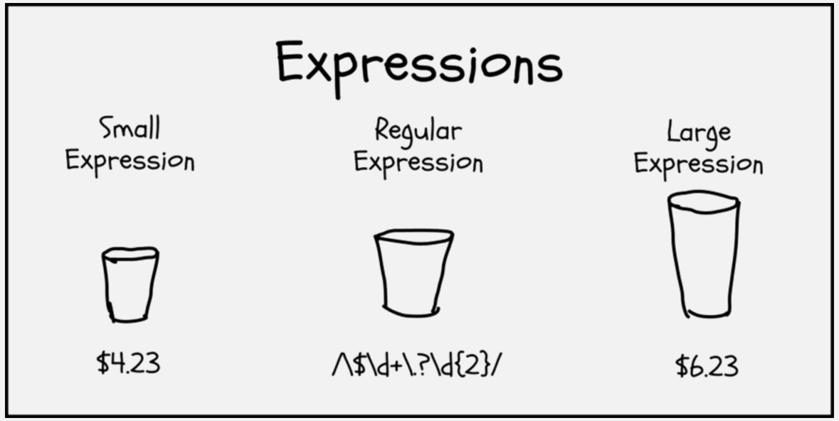
UMB CS622

Regular Expressions

Wednesday September 22, 2021



Announcements

- HW1 graded
  - Use gradescope for grade questions / disputes
- HW2 due Sun 9/26 11:59pm EST

HW1 Review: Inductive Proofs

Must state:

- Induction on what
  - Often, “length of input string”
  - But not always!
- Base Case
- Inductive Case
  - with inductive hypothesis

Every statement and logical step must have justification

Usually taken from:

- Other theorems
- Definitions
- Given assumptions

HW1 Review: Problem 4

Q:

Prove that if some DFA $M = (Q, \Sigma, \delta, q_0, F)$ has a state q such that $\delta(q, a) = q$, for all $a \in \Sigma$, then $\hat{\delta}(q, w) = q$ for all possible strings $w \in \Sigma^*$. Use induction on the length of w.

A:

Claim. If a DFA has a state q such that $\forall a \in \Sigma\ \delta(q, a) = q$, then $\forall w \in \Sigma^*\ \hat{\delta}(q, w) = q$.

Proof. By induction on w.

Basis: Trivially, $\hat{\delta}(q, \epsilon) = q$ by the definition of $\hat{\delta}$.

Induction step: Let $w = w'x$ where $x \in \Sigma$, assume the inductive hypothesis $\hat{\delta}(q, w') = q$. The objective is to show $\hat{\delta}(q, w) = q$ using the claim's precondition $\forall a\ \delta(q, a) = q$.

$$
\begin{aligned}
\hat{\delta}(q, w) &= \hat{\delta}(q, w'x) && \text{by substitution of } w = w'x \\
&= \delta(\hat{\delta}(q, w'), x) && \text{by the definition of } \hat{\delta} \\
&= \delta(q, x) && \text{by the inductive hypothesis} \\
&= q && \text{by the precondition}
\end{aligned}
$$

□

Clearly stated base case and inductive step, with IH

Every logical step has justification

HW1 Review: Problem 3 (part 2)

Q:

Prove that the following language is regular:

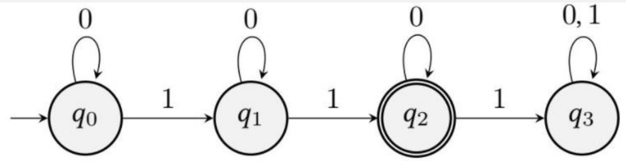
$$\{w \mid w \text{ has exactly two 1s}\}$$

In other words:

1. Design a DFA that recognizes the language; and
2. give an inductive proof that the DFA does indeed recognize the language.

Assume the language contains strings from alphabet $\Sigma = \{0, 1\}$

A:



Claim. $\forall w \in \Sigma^* \; P(w)$, where $P(w) = w \in L(M) \leftrightarrow w \in A$.

Proof. By induction on w.

Basis: $P(\epsilon)$ holds true as $\epsilon \notin L(M)$ (the start state q_0 is not accepting) and $\epsilon \notin A$ (ϵ does not have two 1s).

Induction step: Let $w = w'a$ where $a \in \Sigma$. Assume $P(w')$, and consider $P(w)$ throughout the following case analysis.

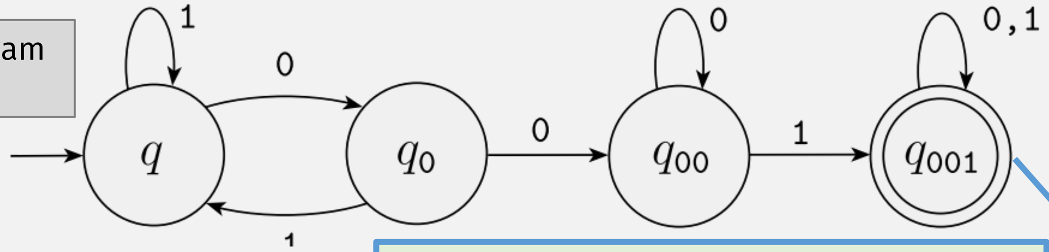
- If w' has zero 1s, then M is in state q_0.
  - Let $a = 0$: M stays in q_0 and rejects with zero 1s.
  - Let $a = 1$: M enters q_1 and rejects with one 1.
- If w' has one 1, then M is in state q_1.
  - Let $a = 0$: M stays in q_1 and rejects with one 1.
  - Let $a = 1$: M enters q_2 and accepts with two 1s.
- If w' has two 1s, then M is in state q_2.
  - Let $a = 0$: M stays in q_2 and accepts with two 1s.
  - Let $a = 1$: M enters q_3 and rejects with three 1s.

Not strong enough!
(needs to say what each state represents)

These need justification
(should come from IH)

So Far: Regular Language Representations

1. State diagram (NFA/DFA)



A practical application: text search … it doesn't fit!

2. Formal description

1. $Q = \{q_1, q_2, q_3\}$,

2. $\Sigma = \{0,1\}$,

3. δ is described as

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2,

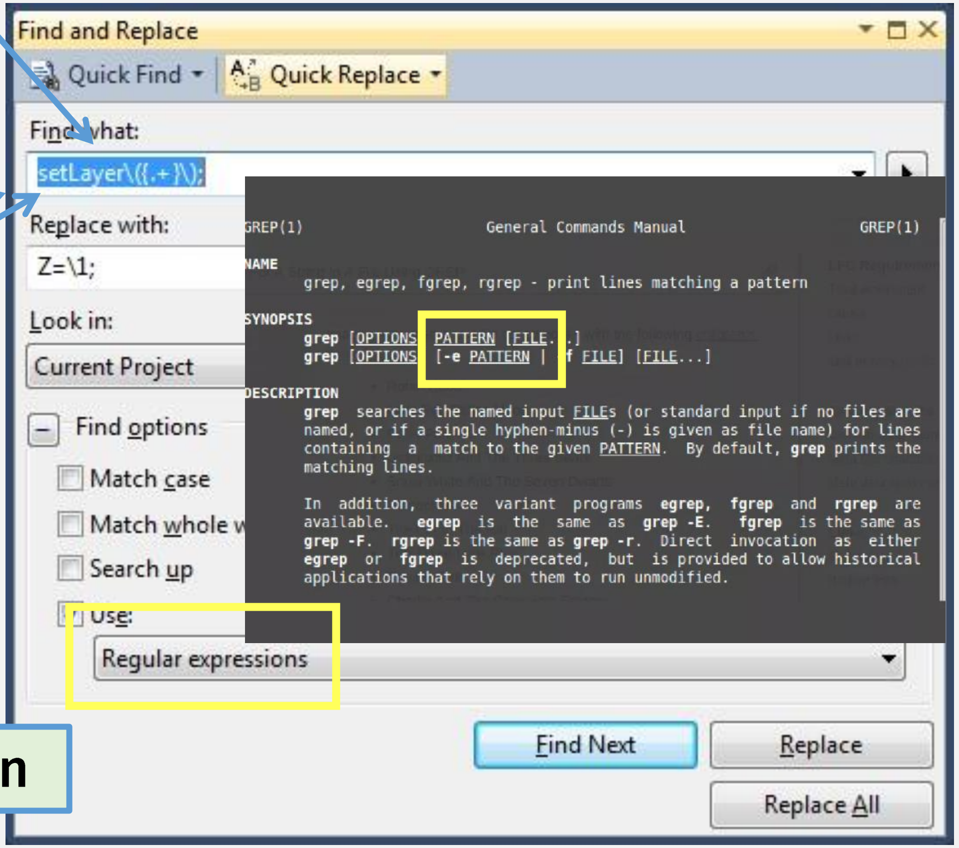
4. q_1 is the start state, and

5. $F = \{q_2\}$.

These define a computer (program) that finds strings containing `001`

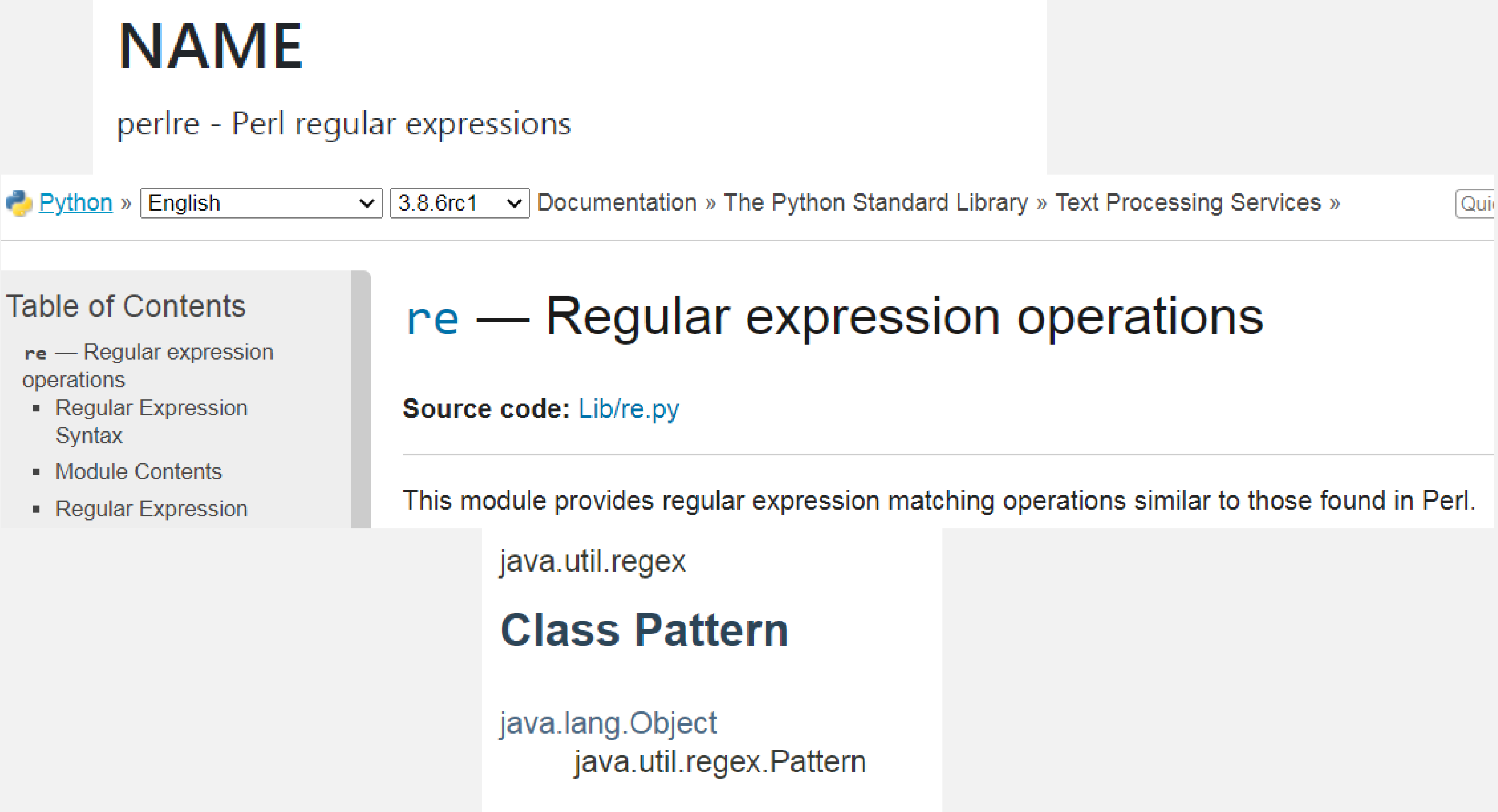
3. $\Sigma^*001\Sigma^*$

Need a more concise notation



Regular Expressions Are Widely Used

- Perl
- Python
- Java
- Every lang!



Regular Expressions: Formal Definition

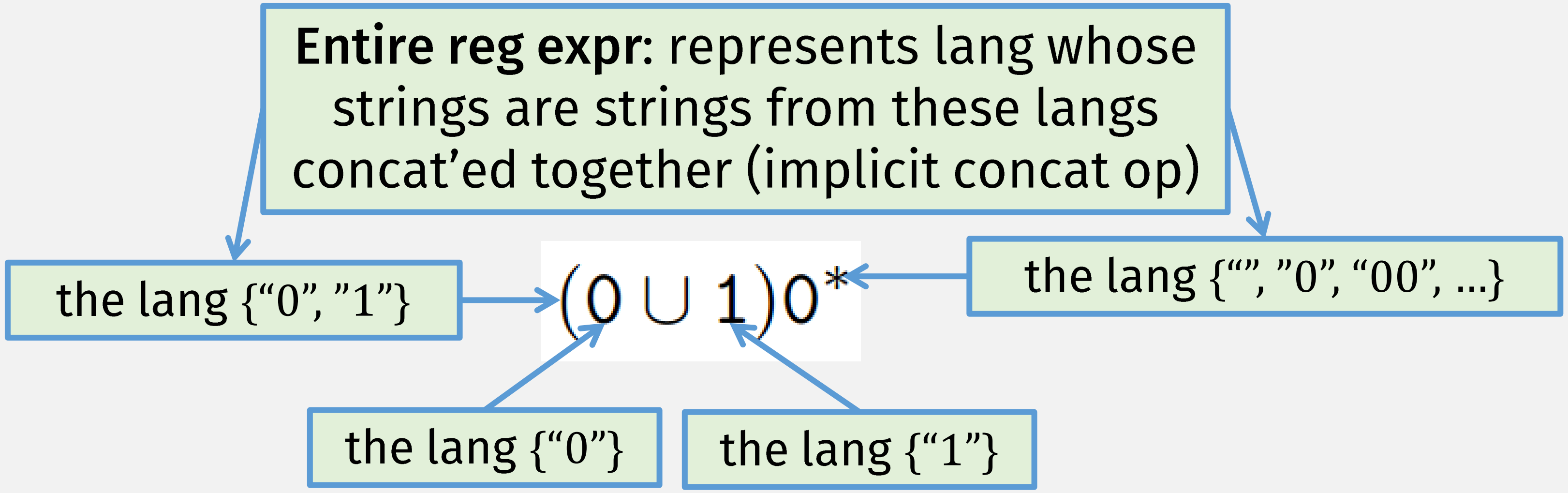
R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ, (A lang containing a) **length-1 string**
2. ε, (A lang containing) **the empty string**
3. $\emptyset$, **The empty set** (i.e., a lang containing no strings)
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions, ← union
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or ← concat
6. (R_1^*), where R_1 is a regular expression. ← star

Base cases plus union, concat, and Kleene star can express <u>any regular language</u>!

(But we have to prove it)

Regular Expression: Concrete Example



- Operator <u>Precedence</u>:
  - Parens
  - Star
  - Concat (sometimes implicit)
  - Union

Thm: A lang is regular **iff** some reg expr describes it

⇒ If a language is regular, it is described by a reg expression

⇐ If a language is described by a reg expression, it is regular

- Easy!
- **For a given regexp, construct the equiv NFA!**
- (we mostly did it already when discussing closed ops)

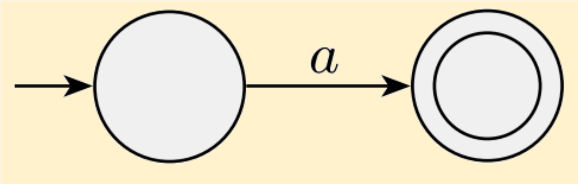
How to show that a lang is regular?

Construct DFA *or* NFA!

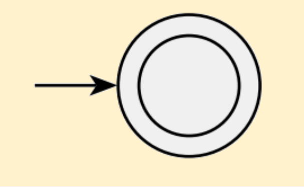
RegExpr→NFA

R is a ***regular expression*** if R is

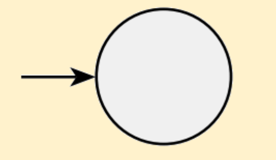
1. a for some a in the alphabet Σ,



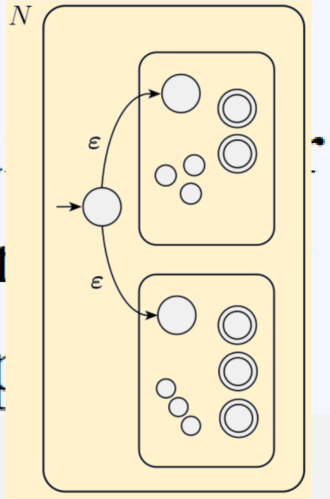
2. ε,

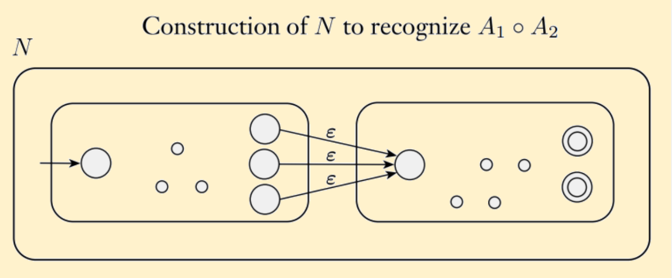


3. $\emptyset$,



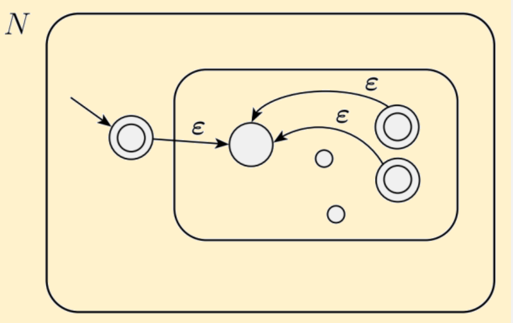
4. $(R_1 \cup R_2)$, where R_1 and R_2 a ... e





5. $(R_1 \circ R_2)$, where R_1 and R_2 ar ... expressions, or

6. (R_1^*), where R_1 is a regular exp



Thm: A lang is regular **iff** some reg expr describes it

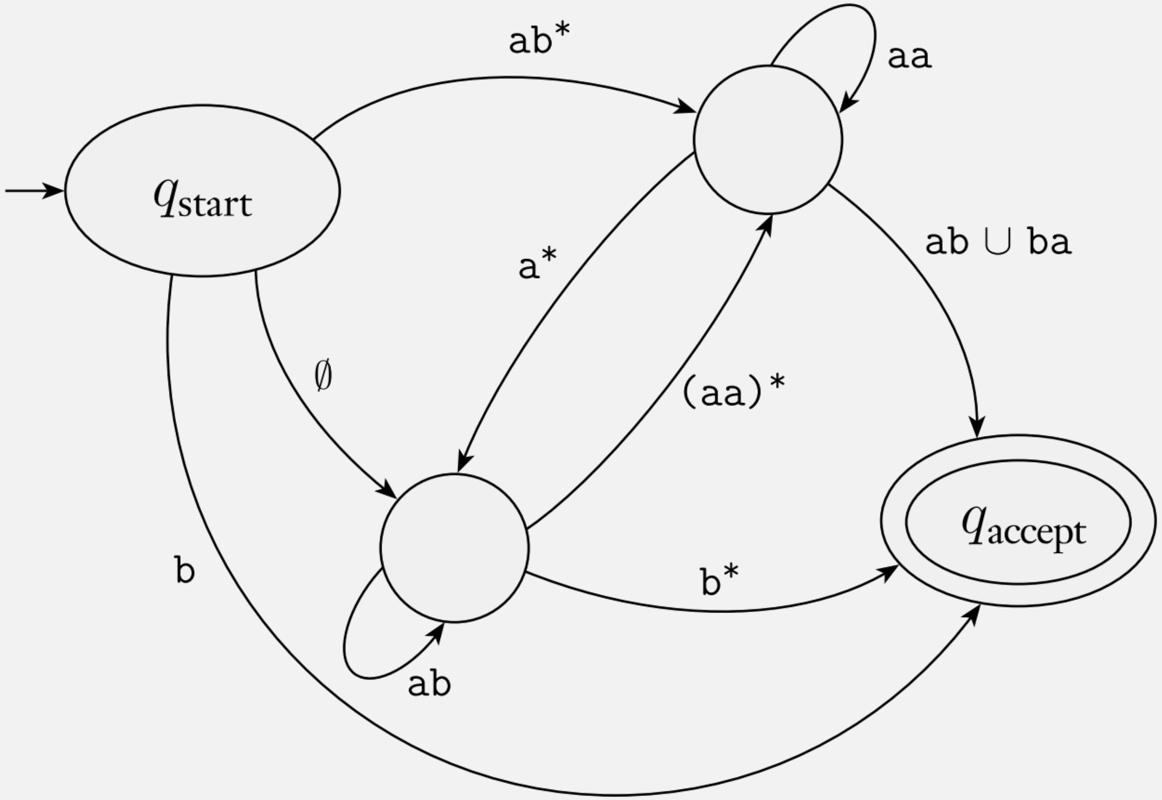
⇒ If a language is regular, it is described by a reg expression

- Harder!
- Need to convert DFA or NFA to Regular Expression
- To do so, need new kind of machine: a GNFA

⇐ If a language is described by a reg expression, it is regular

- Easy!
- Construct the NFA! (**Done**)

Generalized NFAs (GNFAs)



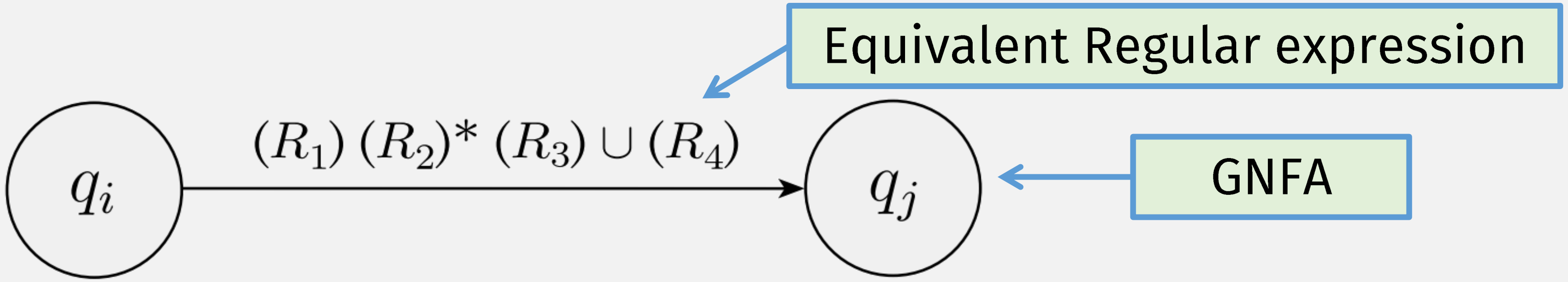
- GNFA = NFA with regular expression transitions

Want to convert GNFAs to Reg Exprs

GNFA→RegExpr function

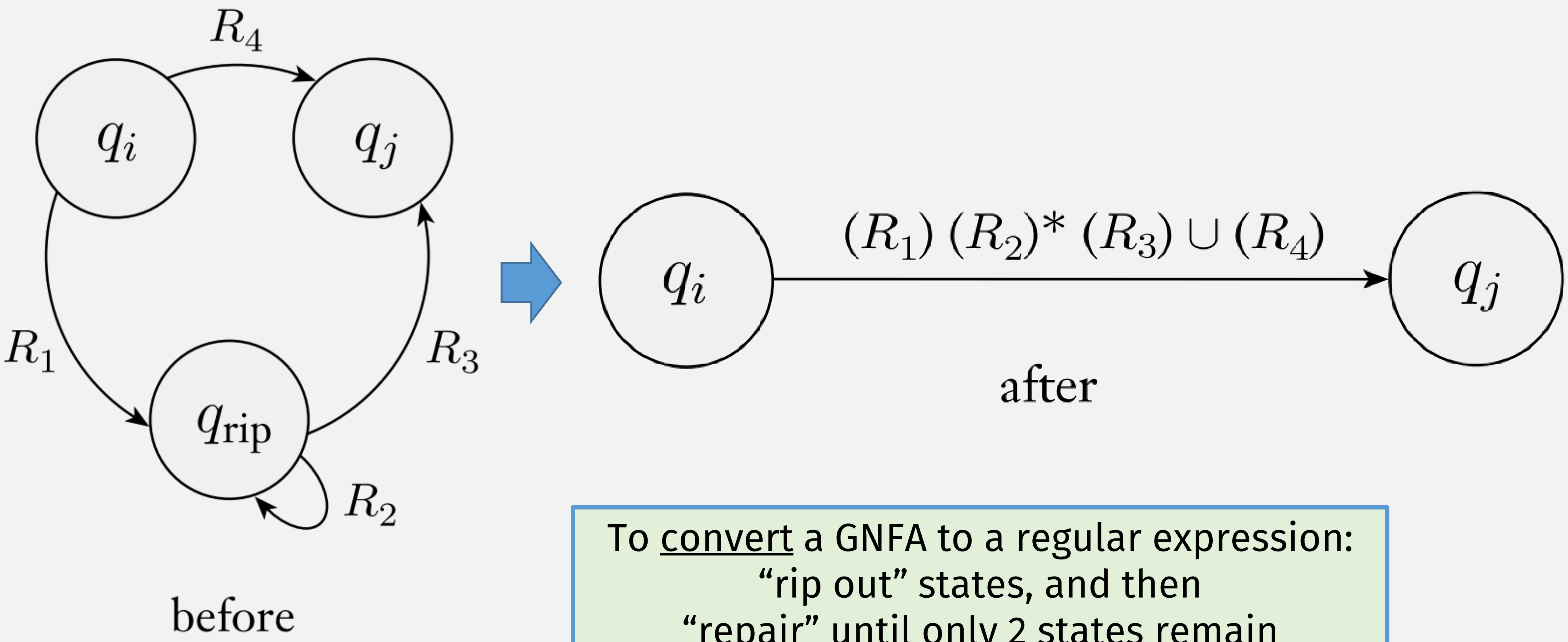
On GNFA input G:

- If G has 2 states, return the regular expression transition, e.g.:



- Else:
  - "Rip out" one state
  - "Repair" the machine to get an equivalent GNFA G'
  - Recursively call **GNFA→RegExpr**(G')

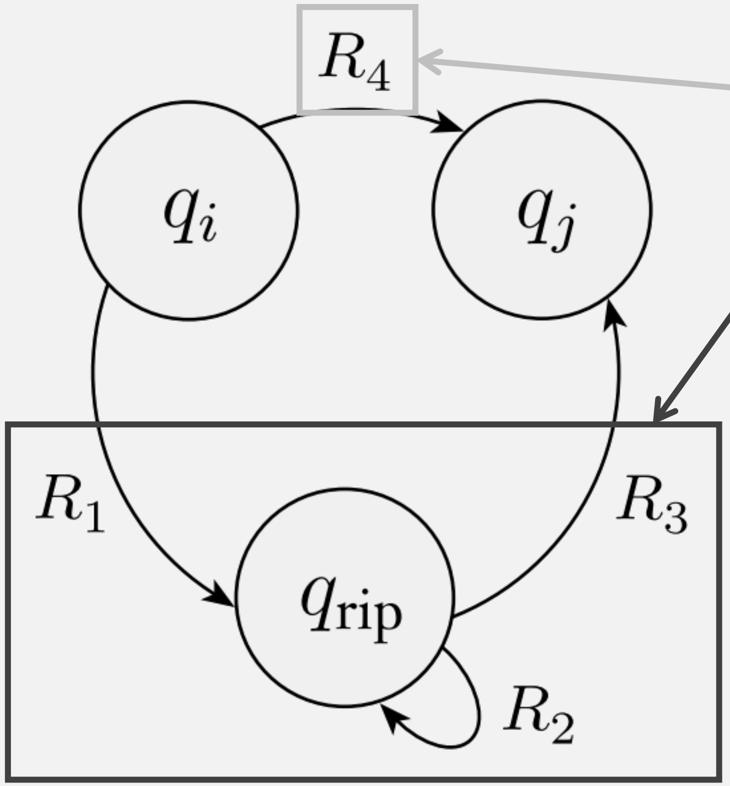
GNFA→RegExpr: "Rip/Repair" step



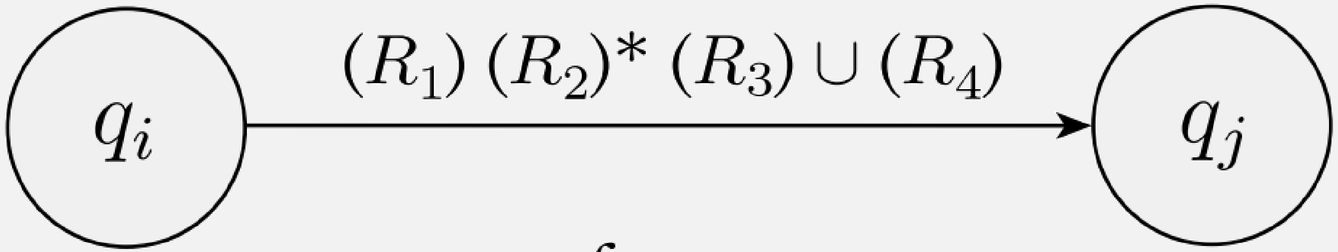
To <u>convert</u> a GNFA to a regular expression:
"rip out" states, and then
"repair" until only 2 states remain

GNFA→RegExpr: "Rip/Repair" step

<u>Before</u>: two paths from q_i to q_j:
1. Not through q_{rip}
2. Through q_{rip}

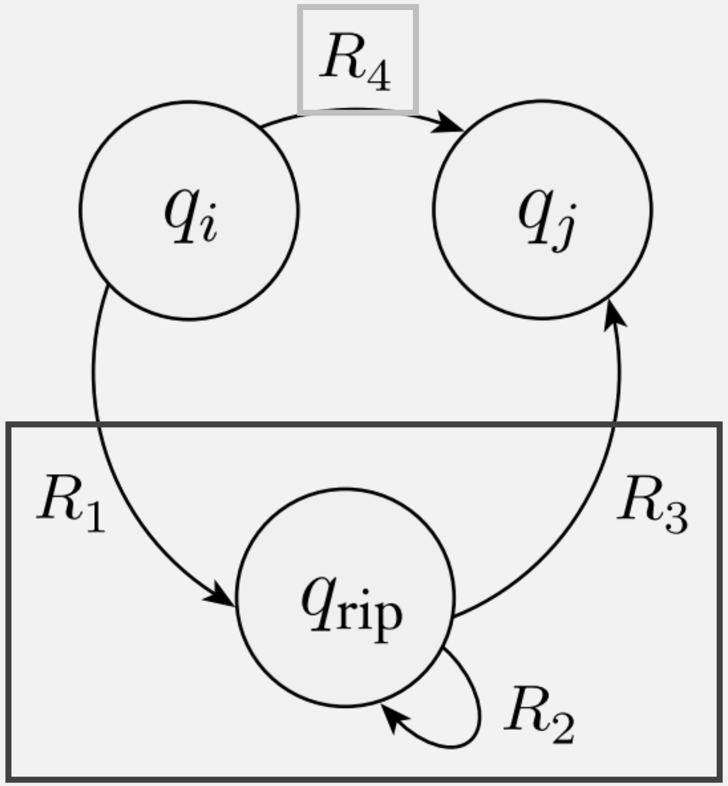


before



after

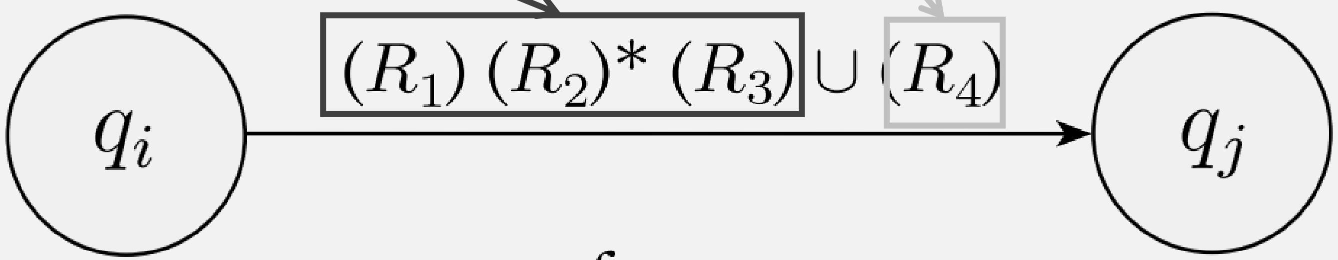
GNFA→RegExpr: “Rip/Repair” step



before

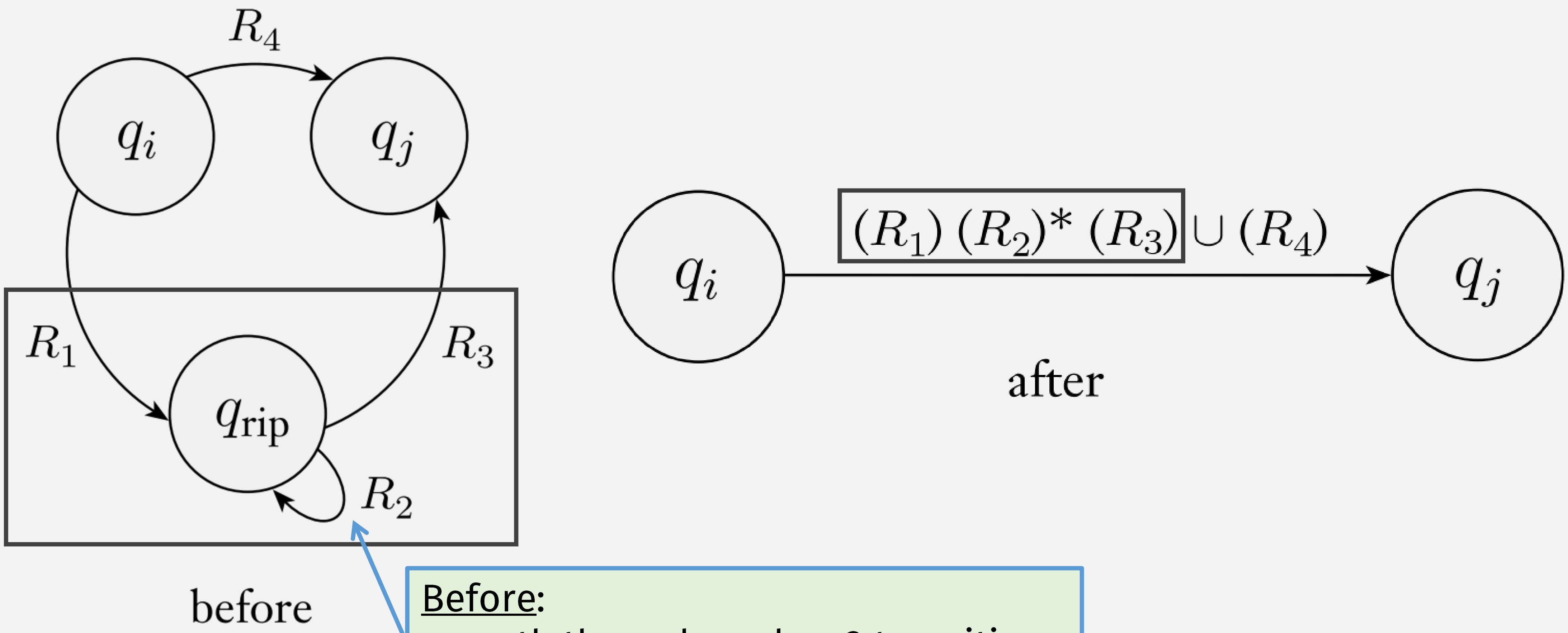
<u>After</u>: still two “paths” from q_i to q_j

1. Not through q_{rip}
2. Through q_{rip}



after

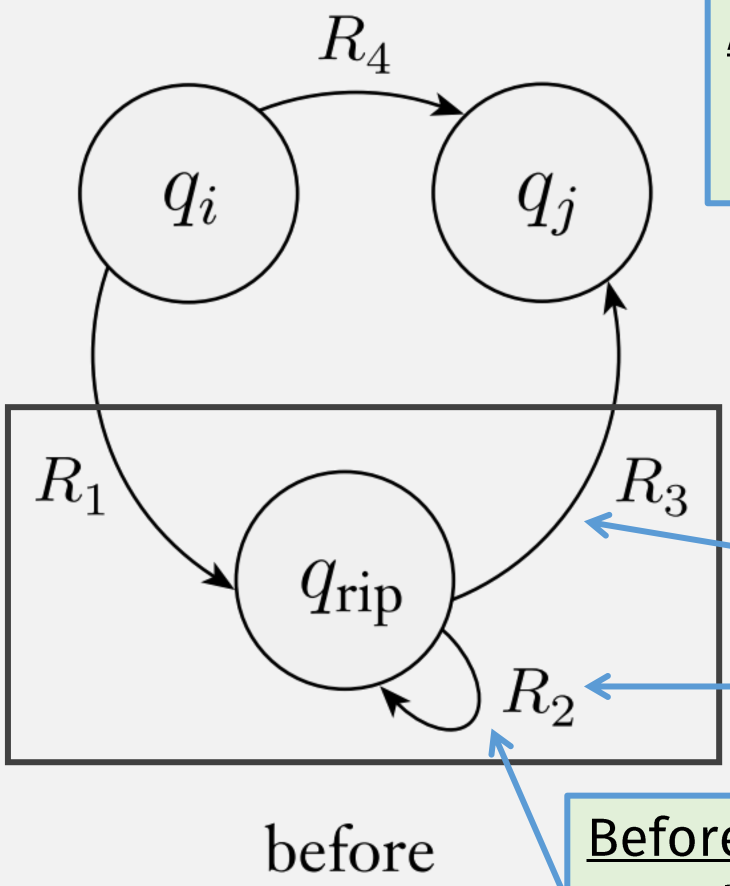
`GNFA→RegExpr`: "Rip/Repair" step

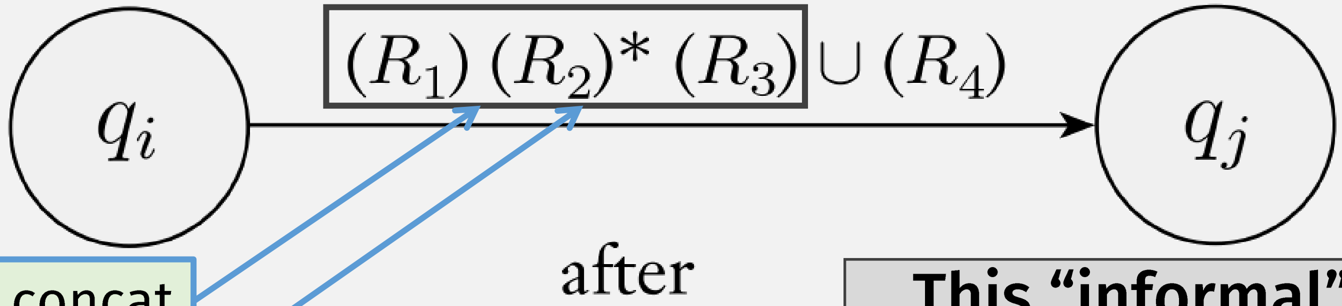


<u>Before</u>:

- path through q_{rip} has 3 transitions
- One is self loop

GNFA→RegExpr: "Rip/Repair" step





<u>After</u>:
- Self loop becomes star operation
- Others are concat'ed together

concat

Star operation

<u>Before</u>:
- path through q_{rip} has 3 transitions
- One is self loop

This "informal" reasoning helps our intuition

Now lets formally prove correctness of GNFA→RegExpr

GNFA→RegExpr “Correctness”

- Where “Correct” means:

Use Proof by induction … on size of G

$$\text{LangOf}(G) = \text{LangOf}(\mathbf{GNFA{\to}RegExpr}(G))$$

This is the property we want to prove

- i.e., **GNFA→RegExpr** must not change the language!

Previously: Recursive (Inductive) Definitions

- Have (at least) two parts:
  - Base case
  - Inductive case
    - Self-reference must be "*smaller*"

This is exactly the structure of an inductive proof!

- <u>Example</u>:

<u>Def</u>: **GNFA→RegExpr**: input G is a GNFA with n states:

Base case — If $n = 2$: return the regular expression on the transition

Inductive case — Else (G has $n > 2$ states):

- "Rip" out one state and "Repair" to get G'
- Recursively Call **GNFA→RegExpr**(G') ← "smaller" self-reference

GNFA→RegExpr is correct

Want to prove: LangOf (G) = LangOf (GNFA→RegExpr(G))

Def: GNFA→RegExpr: input G is a GNFA with n states:

If $n = 2$: return the regular expression on the transition

Else (G has $n > 2$ states):

“Rip” out one state and “Repair” to get G'

Recursively Call **GNFA→RegExpr**(G')

- **Proof** (by induction on size of G):

GNFA→RegExpr is correct

Want to prove: LANGOF (G) = LANGOF (GNFA→RegExpr(G))

Def: **GNFA→RegExpr**: input G is a GNFA with n states:

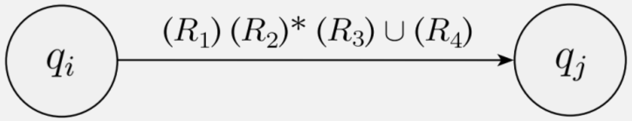
If $n = 2$: return the regular expression on the transition

Else (G has $n > 2$ states):

"Rip" out one state and "Repair" to get G'

Recursively Call **GNFA→RegExpr**(G')

- Proof (by induction on size of G):
  - Base case: G has 2 states
    - LANGOF (G) = LANGOF (**GNFA→RegExpr**(G)) is true, by def of GNFA!

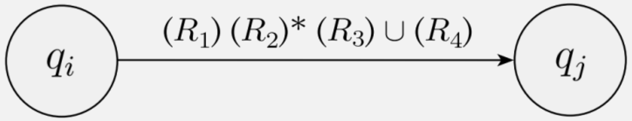


GNFA→RegExpr is correct

Want to prove: LangOf (G) = LangOf (GNFA→RegExpr(G))

Def: **GNFA→RegExpr**: input *G* is a GNFA with *n* states:
- If *n* = 2: return the regular expression on the transition
- Else (*G* has *n* > 2 states):
  - "Rip" out one state and "Repair" to get *G'*
  - Recursively Call **GNFA→RegExpr**(*G'*)

- Proof (by induction on size of *G*):
  - Base case: *G* has 2 states
    - LangOf (*G*) = LangOf (**GNFA→RegExpr**(*G*)) is true!



- IH: Assume LangOf (*G'*) = LangOf (**GNFA→RegExpr**(*G'*))
  - For some *G'* with *n*-1 states

GNFA→RegExpr is correct

Want to prove: LangOf (G) = LangOf (**GNFA→RegExpr**(G))

<u>Def</u>: **GNFA→RegExpr**: input G is a GNFA with n states:

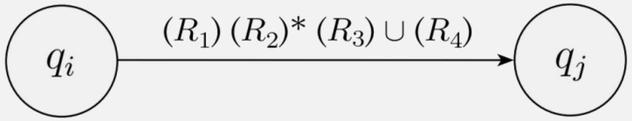
If $n = 2$: return the regular expression on the transition

Else (G has $n > 2$ states):

"Rip" out one state and "Repair" to get G'

Recursively Call **GNFA→RegExpr**(G')

- <u>Proof</u> (by induction on size of G):
  - <u>Base case</u>: G has 2 states
    - LangOf (G) = LangOf (**GNFA→RegExpr**(G)) is true!
  - <u>IH</u>: Assume LangOf (G') = LangOf (**GNFA→RegExpr**(G'))
    - For some G' with <u>n-1</u> states
  - ➢<u>Induction Step</u>: Prove it's true for G with n states



GNFA→RegExpr is correct

Want to prove: LANGOF (G) = LANGOF (GNFA→RegExpr(G))

Def: **GNFA→RegExpr**: input G is a GNFA with n states:

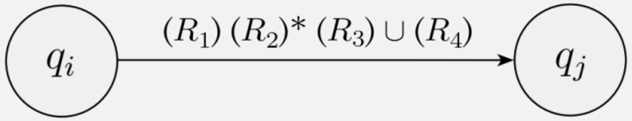
If n = 2: return the regular expression on the transition

Else (G has n > 2 states):

"Rip" out one state and "Repair" to get G'

Recursively Call **GNFA→RegExpr**(G')

- Proof (by induction on size of G):
  - Base case: G has 2 states
    - LANGOF (G) = LANGOF (GNFA→RegExpr(G)) is true!
  - IH: Assume LANGOF (G') = LANGOF (GNFA→RegExpr(G'))
    - For some G' with n-1 states
  - ➢ Induction Step: Prove it's true for G with n states
    - After "rip/repair" step, we have exactly a GNFA G' with n-1 states
    - And we know LANGOF (G') = LANGOF (GNFA→RegExpr(G')) from the IH!



GNFA→RegExpr is correct

Want to prove: LANGOF (G) = LANGOF (GNFA→RegExpr(G))

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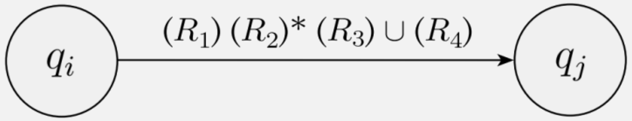
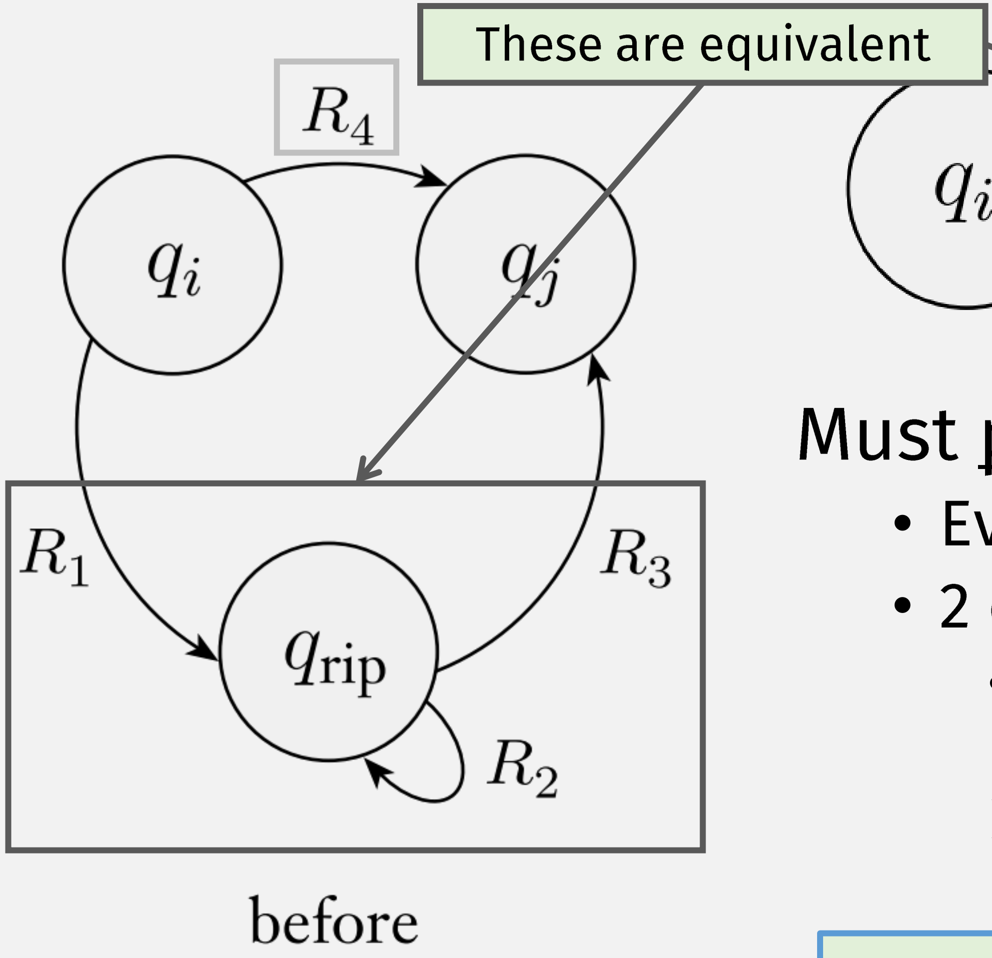
Recursively Call **GNFA→RegExpr**(G')

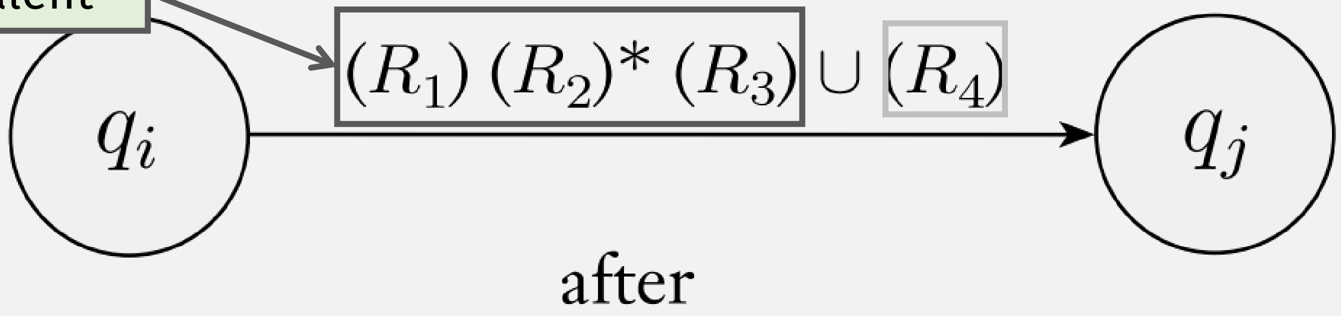
- Proof (by induction on size of G):



  - Base case: G has 2 states
    - LANGOF (G) = LANGOF (GNFA→RegExpr(G)) is true!
  - IH: Assume LANGOF (G') = LANGOF (GNFA→RegExpr(G'))
    - For some G' with n-1 states
  - Induction Step: Prove it's true for G with n states
    - After "rip/repair" step, we have exactly a GNFA G' with n-1 states
    - And we know LANGOF (G') = LANGOF (GNFA→RegExpr(G')) from the IH!
    - ➢ To go from G to G': just need to prove correctness of "rip/repair" step

GNFA→RegExpr: "rip/repair" correctness





Must prove:

- Every string accepted before, is accepted after
- 2 cases:
  - Accepted string does not go through q_{rip}
    - Acceptance unchanged (both use R_4 transition part)
  - ➢ String goes through q_{rip}
    - Acceptance unchanged?

Mostly done this already!
Just need to state more formally

Thm: A lang is regular **iff** some reg expr describes it

⇒ If a language is regular, it is described by a reg expr

- Harder!
- Need to convert DFA or NFA to Regular Expression
- Use **GNFA→RegExpr** to convert GNFA to regular expression! **(Done!)**

⇐ If a language is described by a reg expr, it is regular

- Construct the NFA! (**Done**)

Now we may use regular expressions to represent regular langs.

So a regular language has these equivalent representations:

- DFA
- NFA
- Regular Expression

I.e., we have another way to prove things about reg langs!

Thm: Reverse is Closed for Regular Langs

- For any string $w = w_1 w_2 \cdots w_n$, the ***reverse*** of w, written $w^{\mathcal{R}}$, is the string w in reverse order, $w_n \cdots w_2 w_1$.

  For any language A, let $A^{\mathcal{R}} = \{w^{\mathcal{R}} | \; w \in A\}$

- Theorem: if A is regular, so is $A^{\mathcal{R}}$

- Proof (by induction on regular expressions):

Remember: A language is regular iff it has a regular expression representation

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Case Analysis, assume some regular language A is represented with the regular expression ...

Base cases

1. a for some a in the alphabet Σ, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
2. ε, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
3. $\emptyset$, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular

Inductive cases

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions, ⇐
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Other cases will use similar reasoning

Need to show: if $A_1 \cup A_2$ is a regular language, then $(A_1 \cup A_2)^{\mathcal{R}}$ is regular

"smaller" →

IH: if A_1 and A_2 are the regular languages represented by R_1 and R_2, then $A_1{}^{\mathcal{R}}$ and $A_1{}^{\mathcal{R}}$ are regular too

Proof: $(A_1 \cup A_2)^{\mathcal{R}} = A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}}$, because reversal and union don't affect each other and are interchangeable

$A_1{}^{\mathcal{R}}$ and $A_2{}^{\mathcal{R}}$ are regular (from IH) and union is closed for regular langs (class thm), so $A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}}$ is regular

In-Class quiz 9/22

See gradescope