CS622

NFA <-> DFA

Monday, February 26, 2024

UMass Boston CS

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

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Announcements

- HW 3 out
    - Due Mon 3/4 12pm EST (noon)
- HW 1 grades returned
- Use Gradescope re-grade request for all questions / complaints!

Previously

Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$

Is Concatenation Closed?

THEOREM

The class of regular languages is closed under the concatenation operation.

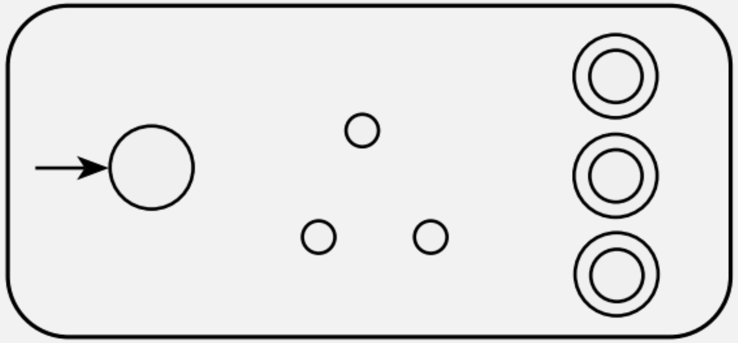
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof requires: Constructing new machine

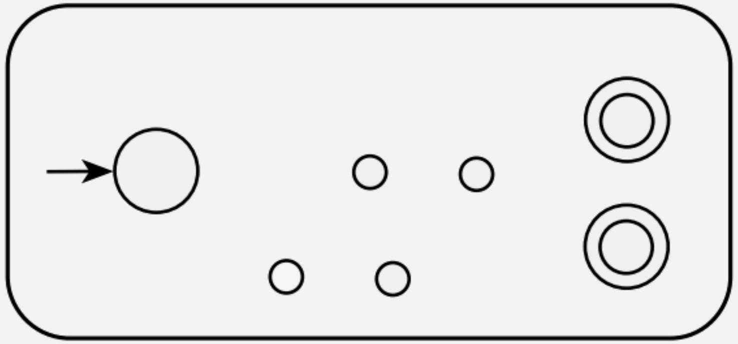
- How does it know when to switch machines?
  - Can only read input once

Concatentation

M_1



M_2



Let M_1 recognize A_1, and M_2 recognize A_2.

Want: Construction of N to recognize $A_1 \circ A_2$

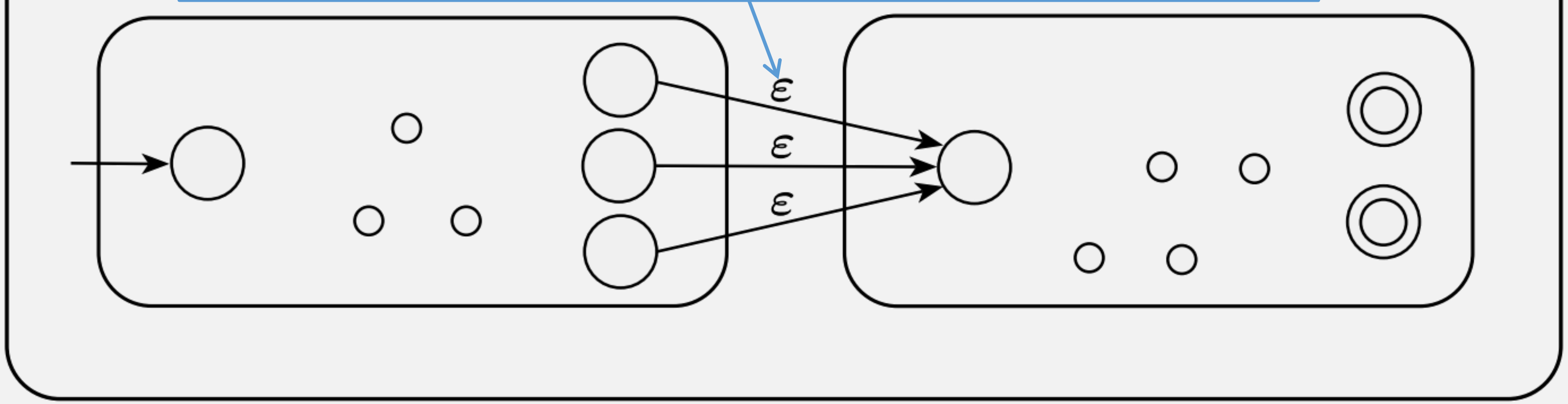
N is an **NFA**! It can:
- Keep checking 1st part with M_1

and

- Move to M_2 to check 2nd part

N

ε = "empty transition" = reads no input
Allows N to be in both machines at the same time!



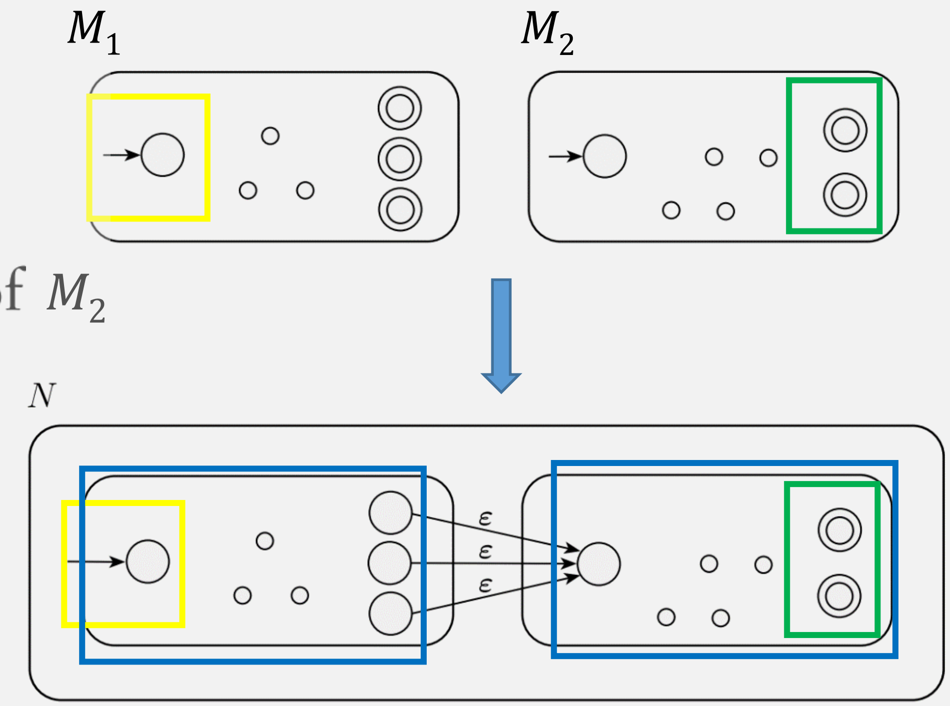
Concatenation is Closed for Regular Langs

PROOF (part of)

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,



Concatenation is Closed for Regular Langs

Wait, is this true?

PROOF (part of)

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$

2. The state q_1 is the same as the start state of M_1

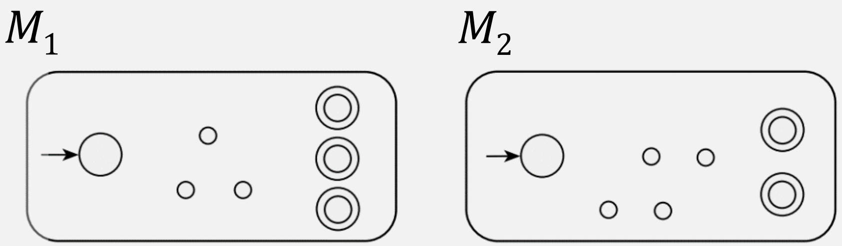
3. The accept states F_2 are the same as the accept states of M_2

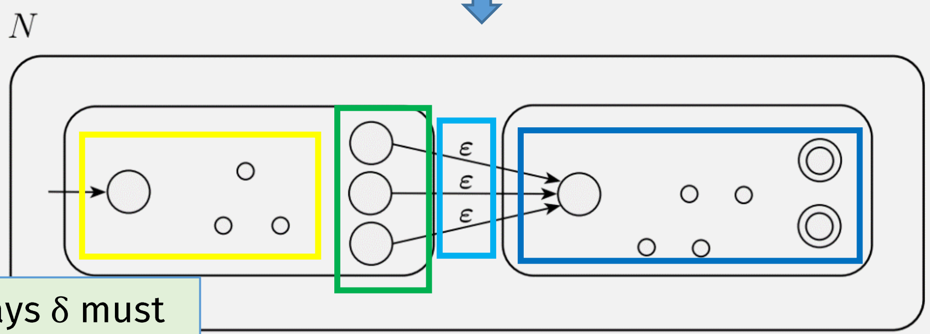
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \{\delta_1(q, a)\} \;?\; & q \in Q_1 \text{ and } q \notin F_1 \\ \{\delta_1(q, a)\} \;?\; & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \{\delta_2(q, a)\} \;?\; & q \in Q_2. \end{cases}$$

And: $\delta(q, \varepsilon) = \emptyset$, for $q \in Q, q \notin F_1$

NFA def says δ must map every state and ε to set of states





??? ∎

Is Concat Closed For Regular Langs?

Proof?

Statements	Justifications
1. A_1 and A_2 are regular languages	1. Assumption
2. A DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes A_1	2. Def of Reg Lang (Coro)
3. A DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes A_2	3. Def of Reg Lang (Coro)
4. Construct NFA $M = (Q, \Sigma, \delta, q_0, F)$ ☑	4. Def of NFA
5. M recognizes ~~$A_1 \cup A_2$~~ $A_1 \circ A_2$	5. See Examples Table
6. ~~$A_1 \cup A_2$~~ $A_1 \circ A_2$ is a regular language	6. ??? Does NFA recognize reg langs?
7. The class of regular languages is closed under t concatenation n operation. In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.	7. From stmt #1 and #6

Q.E.D.?

Previously

A DFA's Language

- For **DFA** $M = (Q, \Sigma, \delta, q_0, F)$
- M **accepts** w if $\hat{\delta}(q_0, w) \in F$
- M **recognizes** language $\{w | M \text{ accepts } w\}$

Definition: A DFA's language is a **regular language**

An NFA's Language?

- For **NFA** $N = (Q, \Sigma, \delta, q_0, F)$

Intersection … … with accept states …

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$ … is not empty set
  - i.e., accept if final states contains at least one accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

Q: What kind of languages do NFAs recognize?

Concatenation Closed for Reg Langs?

- **Combining DFAs to recognize concatenation of languages …**

  … produces an **NFA**

- So **to prove concatenation is closed …**

  … we must prove that NFAs *also* recognize regular languages.

Specifically, we must prove:

NFAs ⇔ regular languages

“If and only if” Statements

$X \Leftrightarrow Y$ = “*X* if and only if *Y*” = *X* iff Y = *X* <=> *Y*

Represents <u>two</u> statements:

1. $\Rightarrow$ if *X*, then *Y*
   - “**forward**” direction

2. $\Leftarrow$ if *Y*, then *X*
   - “**reverse**” direction

How to Prove an "iff" Statement

$X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y

Proof has <u>two</u> (If-Then proof) parts:

1. $\Rightarrow$ if X, then Y
   - "**forward**" direction
   - assume X, then use it to prove Y
2. $\Leftarrow$ if Y, then X
   - "**reverse**" direction
   - assume Y, then use it to prove X

Proving NFAs Recognize Regular Langs

Theorem:
A language L is regular **if and only if** some NFA N recognizes L.

Proof: 2 parts

⇒ If L is regular, then some NFA N recognizes it.
(Easier)
- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 3)

⇐ If an NFA N recognizes L, then L is regular.

Full Statements & Justifications?

"equivalent" = "recognizes the same language"

⇒ If L is regular, then some NFA N recognizes it

Statements	**Justifications**
1. L is a regular language	1. Assumption
2. A DFA M recognizes L	2. Def of Regular lang (Coro)
3. Construct NFA $N = \texttt{convert}(M)$	3. See hw ~~2~~ 3!
4. DFA M is **equivalent** to NFA N	4. See Equiv. table!
5. An NFA N recognizes L	5. ???
6. If L is a regular language, then some NFA N recognizes it	6. By Stmts #1 and # 5

Assume the "if" part …

… use it to prove "then" part

"Proving" Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$

NFA $N =$ convert(M)

$\hat{\delta}(q_0, w) \in F$ for some string w

Note:
extra column

String	M accepts?	N accepts?	N accepts? Justification
w	Yes	???	See justification #1
w'	No	???	See justification #2?
…			

If M accepts w …

Then we know …

There is some sequence of states: $r_1 \ldots r_n$, where $r_i \in Q$ and

$$r_1 = q_0 \text{ and } r_n \in F$$

Then N accepts?/rejects? w because …

Justification #1?
There is an accepting sequence of set of states in N … for string w

"Proving" Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$
NFA $N = \text{convert}(M)$
$\hat{\delta}(q_0, w) \in F$ for some string w
$\hat{\delta}(q_0, w') \notin F$ for some string w'

String	M accepts?	N accepts?	N accepts? Justification
w	Yes	???	See justification #1
w'	No	???	See justification #2?
…			

If M rejects w' …

Then we know …

Then N accepts?/rejects? w' because …

Justification #2?

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

☑ ⇒ If L is regular, then some NFA N recognizes it.

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 3)

⇐ If an NFA N recognizes L, then L is regular.

(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA N → an equivalent DFA

"equivalent" =
"recognizes the same language"

How to convert **NFA→DFA**?

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
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A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
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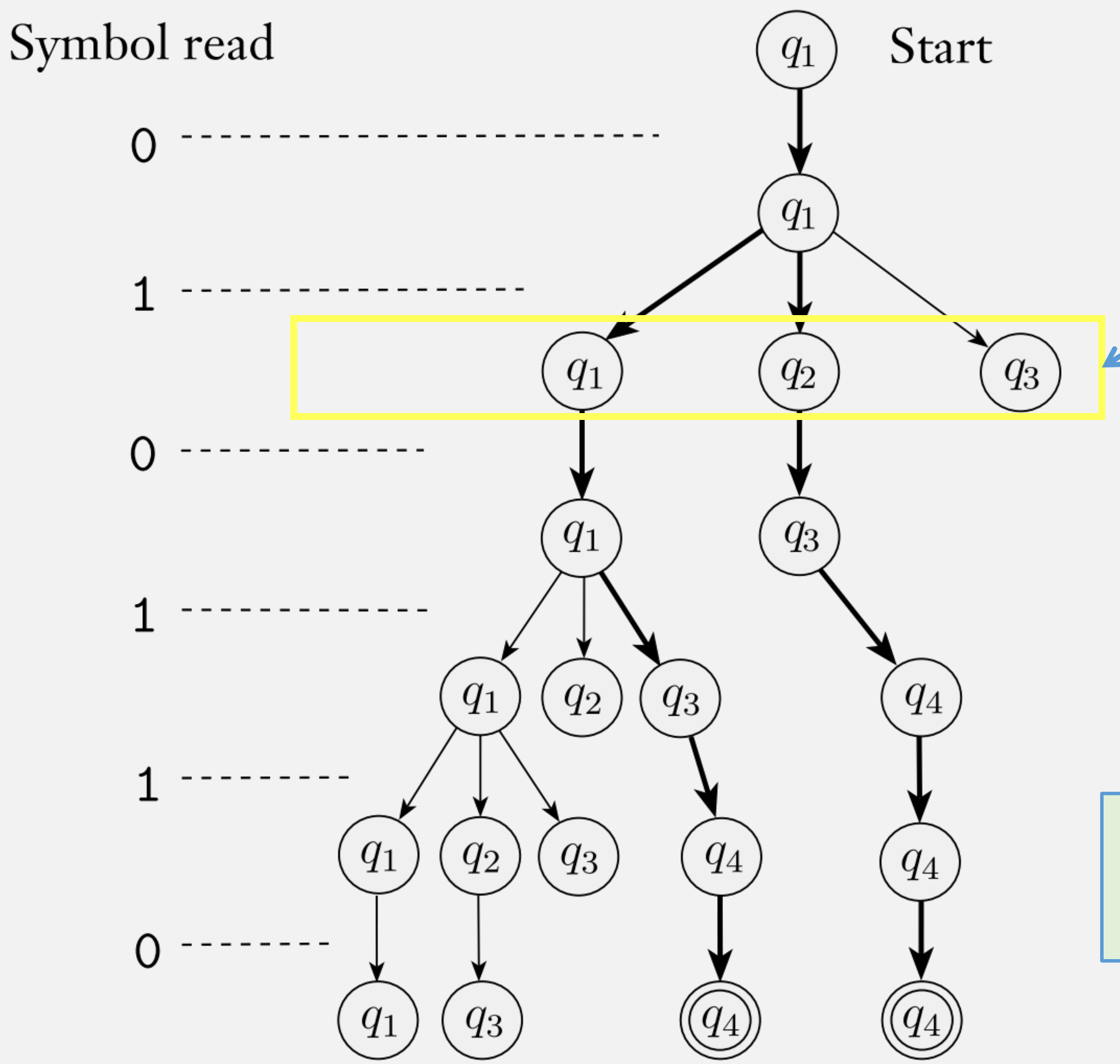
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA
= set of states in the NFA



NFA computation can be in multiple states

DFA computation can only be in one state

So **encode:**
a set of NFA states
as one DFA state

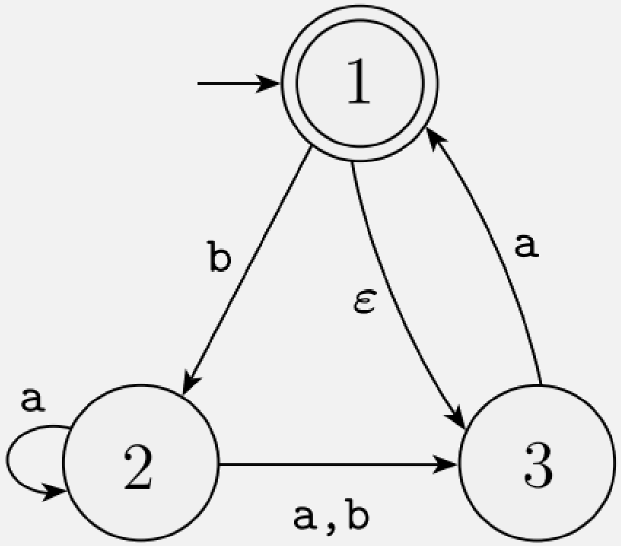
This is similar to the proof strategy from
"Closure of union" where:
a state = a pair of states

Convert **NFA→DFA**, Formally

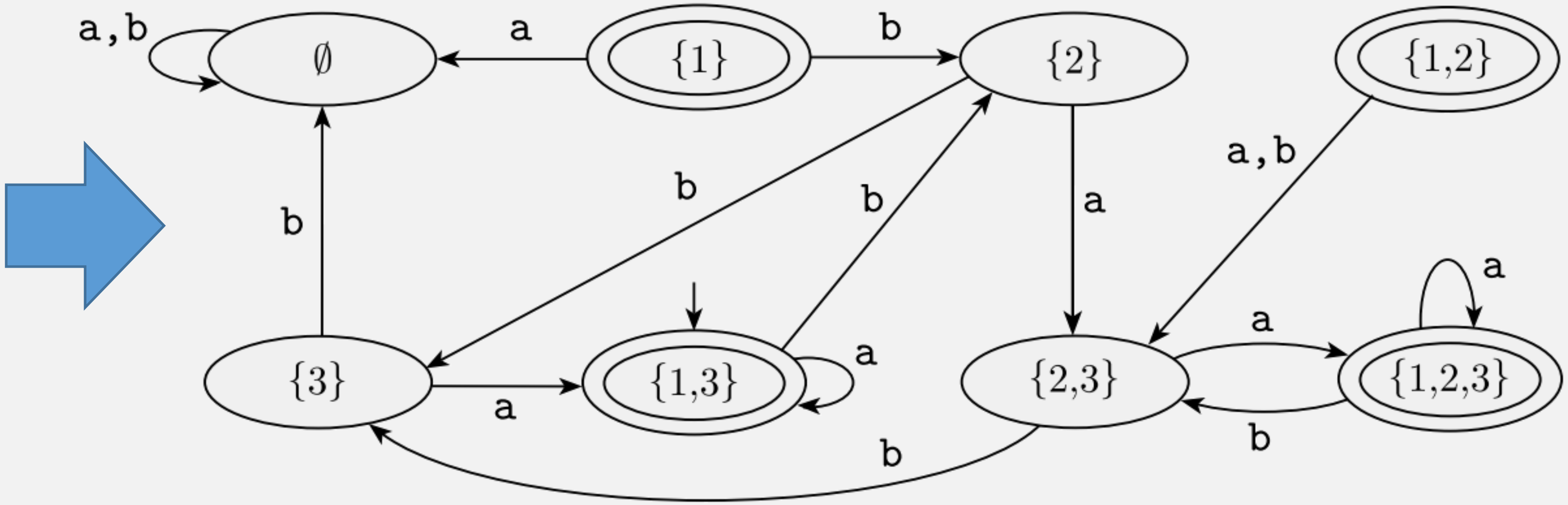
- Let **NFA** N = $(Q, \Sigma, \delta, q_0, F)$
- **An equivalent DFA** M **has states** Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:

- Let NFA $N_4 = (Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA D has states $= \mathcal{P}(Q)$ (power set of Q)



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$

   A DFA state = a set of NFA states

   qs = DFA state = set of NFA states

2. For $qs \in Q_{DFA}$ and $a \in \Sigma$
   - $\delta_{DFA}(qs, a) = \bigcup_{q \in qs} \delta_{NFA}(q, a)$

   A DFA step = an NFA step for all states in the set

3. $q_{0DFA} = \{q_{0NFA}\}$

4. $F_{DFA} = \{ qs \in Q_{DFA} \mid qs \text{ contains accept state of } N \}$

Flashback: Adding Empty Transitions

- Define the set $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

- Recursive case:

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

Almost the same, except …

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$

2. For $qs \subseteq Q_{DFA}$ and $a \in \Sigma$
   - $\delta_{DFA}(qs, a)$ ~~$= \bigcup_{q \in qs} \delta_{NFA}(q, a)$~~

   $S = \bigcup_{q \in qs} \delta_{NFA}(q, a)$

   $$\bigcup_{s \in S} \varepsilon\text{-REACHABLE}(s)$$

3. $q_{0DFA} =$ ~~$\{q_{0NFA}\}$~~ $\varepsilon\text{-REACHABLE}(q_{0NFA})$

4. $F_{DFA} = \{\, qs \in Q_{DFA} \mid qs$ contains accept state of $N \,\}$

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Statements & Justifications?

Proof:

⇒ If L **is regular,** then **some NFA** N **recognizes it.**

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 3)

⇐ If **an NFA** N **recognizes** L, then L **is regular.**

Examples table?

(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA N → an equivalent DFA …

… using our NFA to DFA algorithm! ∎

Concatenation is Closed for Regular Langs ☑

Wait, is this true?

If a language has an **NFA** recognizing it, then **it** is a **regular** language

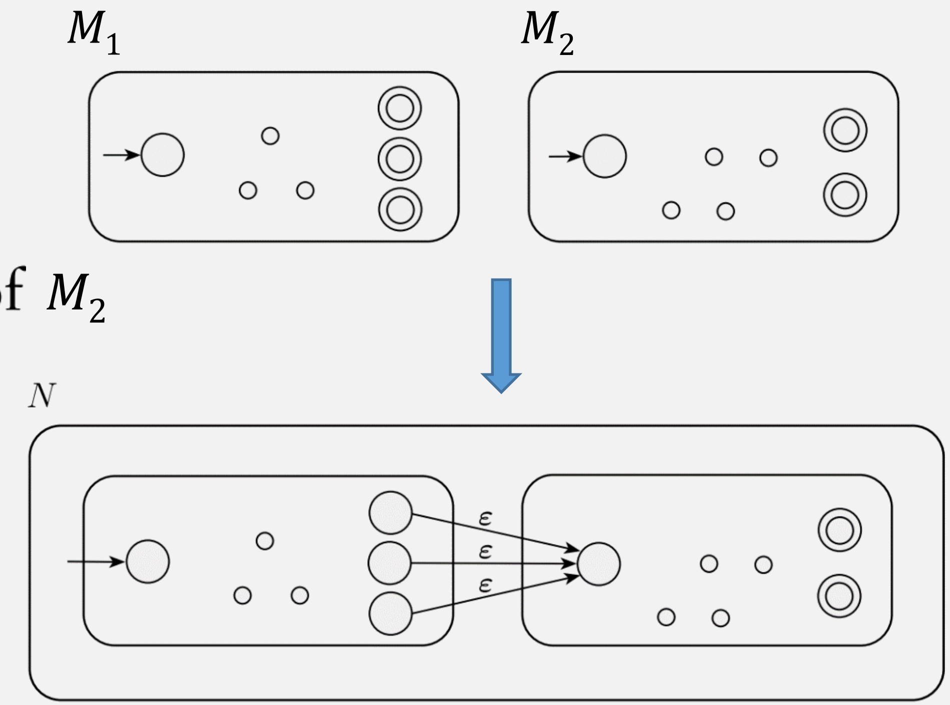
PROOF

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \{\delta_1(q,a)\} & q \in Q_1 \text{ and } q \notin F_1 \\ \{\delta_1(q,a)\} & q \in F_1 \text{ and } a \neq \varepsilon \\ \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \{\delta_2(q,a)\} & q \in Q_2. \end{cases}$$



And: $\delta(q, \varepsilon) = \emptyset$, for $q \in Q, q \notin F_1$??? ■ ☑

New possible proof strategy!

Concat Closed for Reg Langs: Use NFAs Only

PROOF

If language is **regular**, then **it has an NFA** recognizing it …

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and

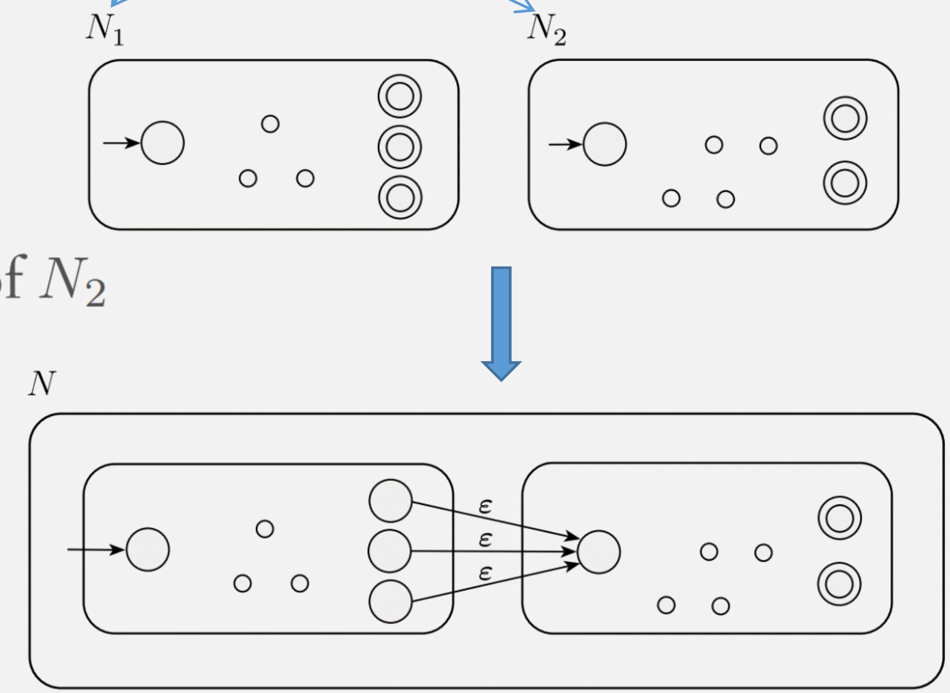
NFAs $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

????

$$\delta(q,a) = \begin{cases} \delta_1(q,a) \; ? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) \; ? & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) \; ? & q \in Q_2. \end{cases}$$



Union: $A \cup B = \{x | \ x \in A \text{ or } x \in B\}$

Flashback: Union is Closed For Regular Langs

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Combine the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Flashback: Union is Closed For Regular Langs

Proof

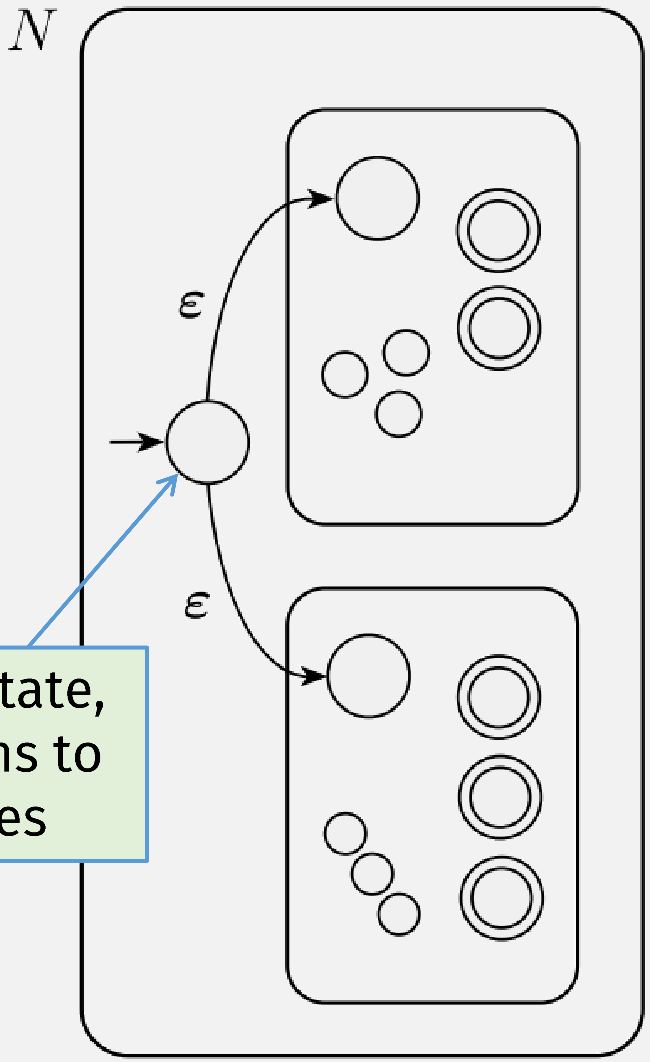
- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: **a new machine** $M = (Q, \Sigma, \delta, q_0, F)$ **using** M_1 **and** M_2
- states of M: $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2
- M transition fn: $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
- M start state: (q_1, q_2)
- M accept states: $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

State in M = M_1 state + M_2 state

M step = a step in M_1 + a step in M_2

Accept if either M_1 or M_2 accept

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

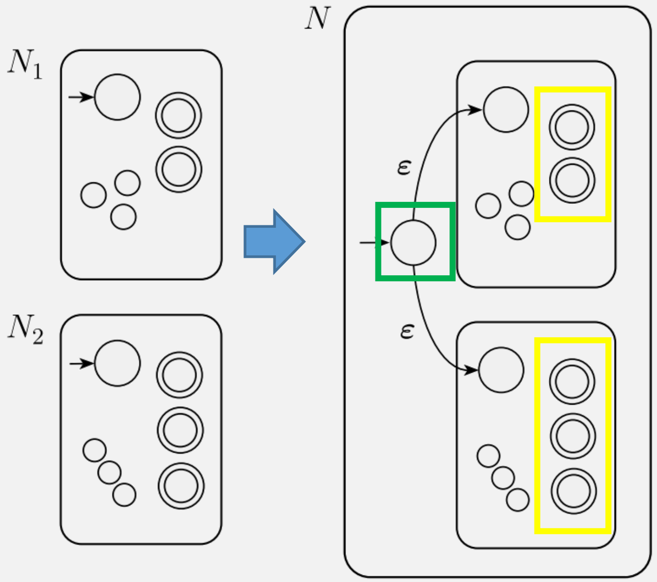
Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.



Union is Closed for Regular Languages

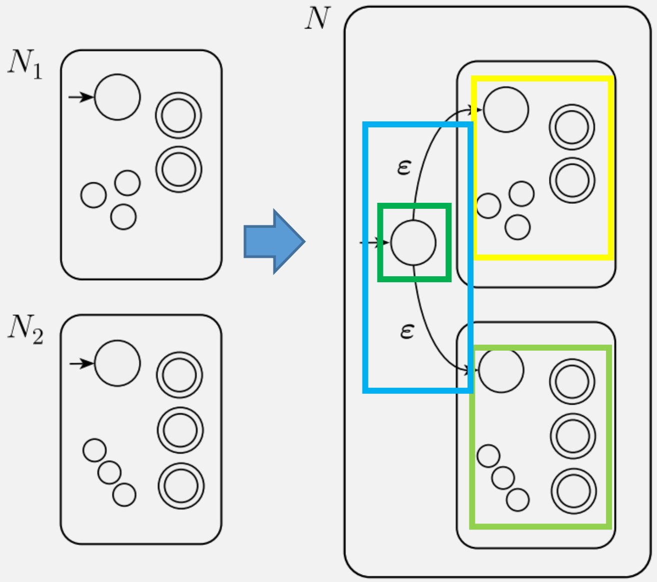
PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) \; ? & q \in Q_1 \\ \delta_2(q, a) \; ? & q \in Q_2 \\ \{q_1, q_2\} \; ? & q = q_0 \text{ and } a = \varepsilon \\ \emptyset \; ? & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$



Don't forget Statements and Justifications!

List of Closed Ops for Reg Langs (so far)

- ☑ Union
- ☑ Concatentation
- Kleene Star (repetition) ?

Star: $A^* = \{x_1x_2\ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

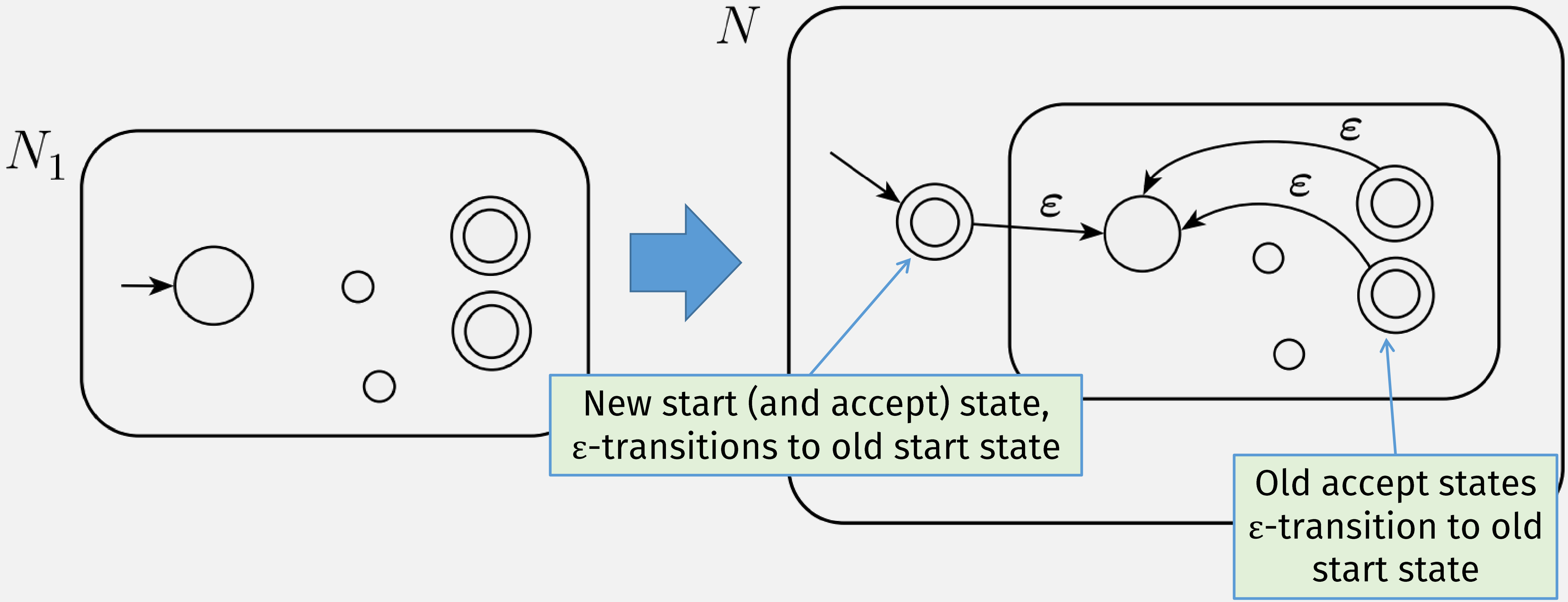
If $A = \{\mathtt{good}, \mathtt{bad}\}$

$$A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad}, \ldots\}$$

Note: repeat zero or more times

(this is an infinite language!)

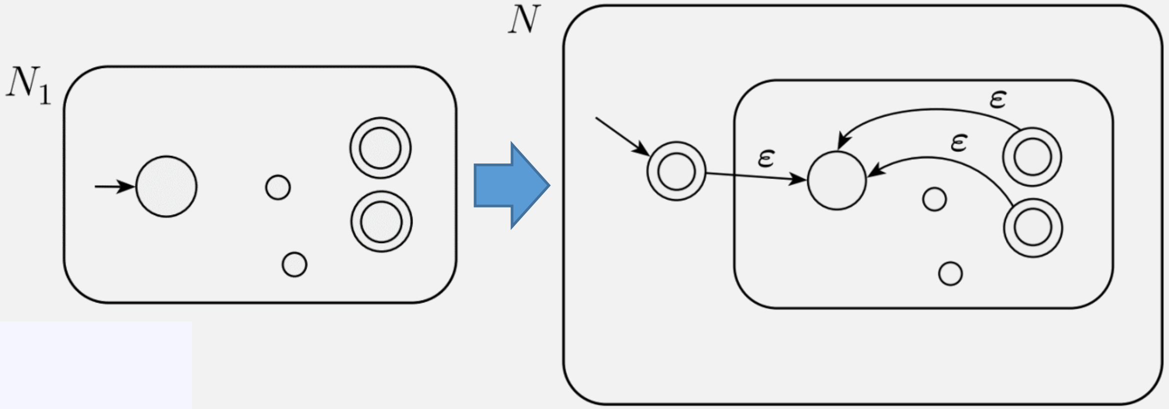
Kleene Star



N_1
N
ε
ε
ε
New start (and accept) state,
ε-transitions to old start state
Old accept states
ε-transition to old
start state

In-class exercise

Kleene Star is Closed for Regular Langs



THEOREM

The class of regular languages is closed under the star operation.

In-class solution

Kleene Star is Closed for Regular Langs

(part of)

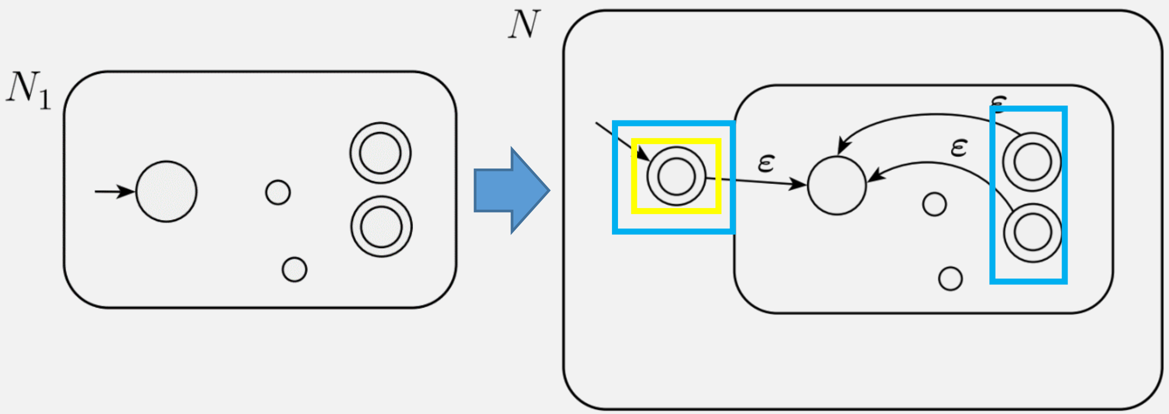
PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

1. $Q = \{q_0\} \cup Q_1$

2. The state q_0 is the new start state.

3. $F = \{q_0\} \cup F_1$

Kleene star of a language must accept the empty string!



In-class solution

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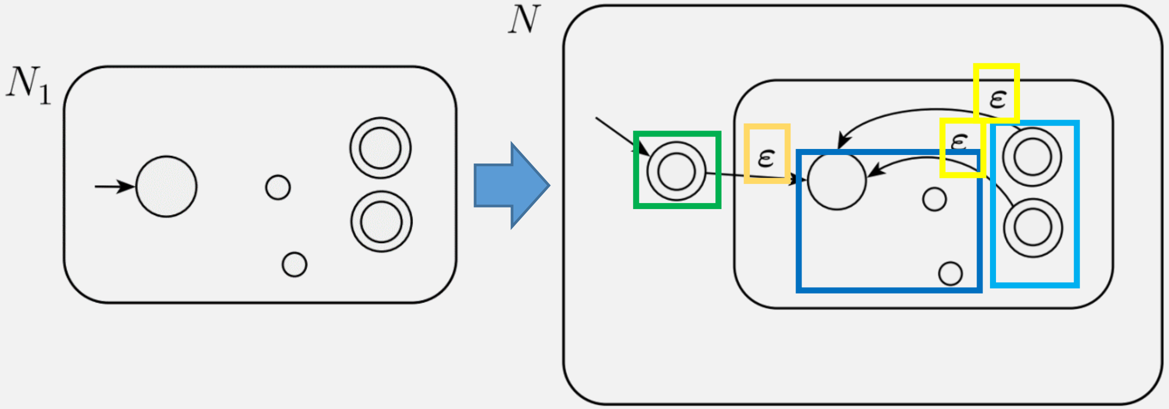

1. $Q = \{q_0\} \cup Q_1$
2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a)\,? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a)\,? & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a)\,? \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} \quad ? & q = q_0 \text{ and } a = \varepsilon \\ \emptyset \quad ? & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Next Time: Why These Closed Operations?

- Union
- Concat
- Kleene star

All regular languages can be constructed from:
- single-char strings, and
- these three combining operations!

Submit in-class work 2/26

On gradescope