Non-Regular Languages

Sept 30, 2020

HW3

- Due in 12 days (Sun Oct 11, 11:59pm EST)
- Last assignment with coding (for a while at least ☺)
- Really fun!

HW2 presentations

Oscar (python)

Francisco (java)

Ivana (python)

So … XML is not a regular language

- How do we know?

- In general, we have many ways to show a lang is regular
  - Construct DFA or NFA
  - Create regular expressions

- But how do we show that a language is not regular???

Recall: Designing DFAs or NFAs

- States = the machine's **memory**!
  - Each state "stores" some information
  - Finite states = finite amount of memory
  - And must be allocated in advance

- Can't do this with input:

00111100001101010

Interpret as a number "n"

Accept if this part has "n" zeroes

A non-regular language

- L = { 0^n1^n | n >= 0 }
- A DFA recognizing L would require infinite states! (impossible)
- This lang is the essence of XML!
  - To better see this replace “0” -> “<tag>“ and “1” -> “</tag>”
- The problem is the **nestedness**
  - Regular langs cannot keep track of arbitrary nestedness
  - So most programming langs are also not regular!

How to prove a language is <u>not</u> regular?

The "Pumping" Lemma

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

What the heck???

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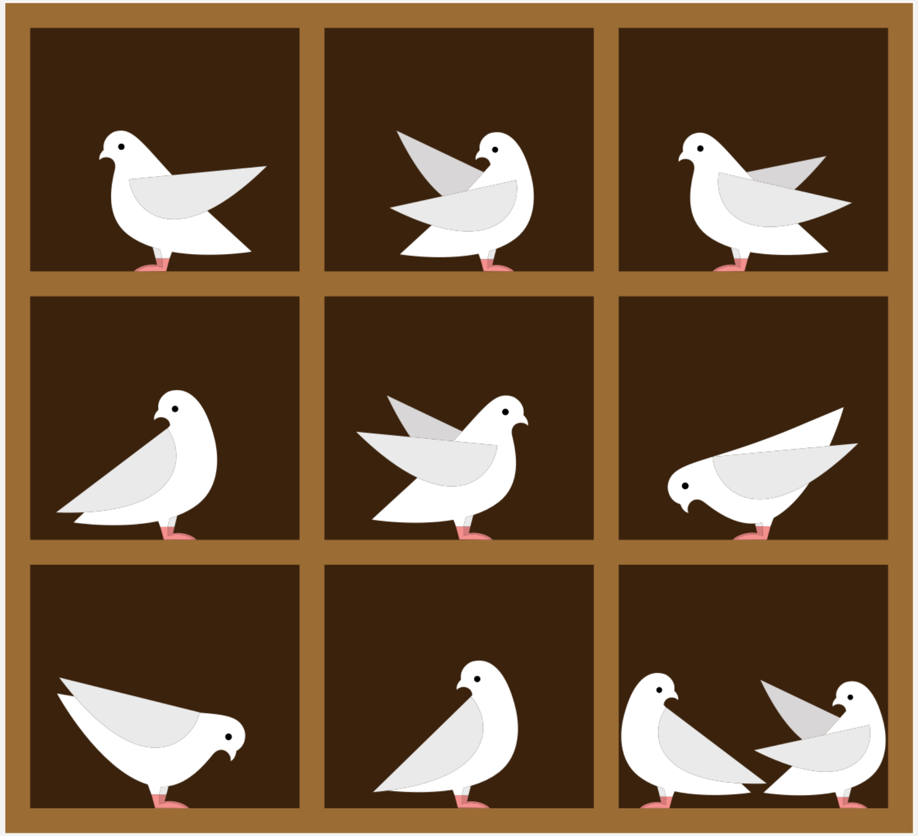
- What is the pumping length saying? (you get to choose p!)
- Langs that are finite, e.g., {“ab” , “cd”} or { } are obviously regular
  - Just choose pumping length > longest string
- Only infinite languages are interesting!
  - Pumping length p >= num states: guarantees repeated states

The Pigeonhole Principle



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- "Long enough" strings must repeat, since there are only finite states.
  - "Pigeonhole principle"

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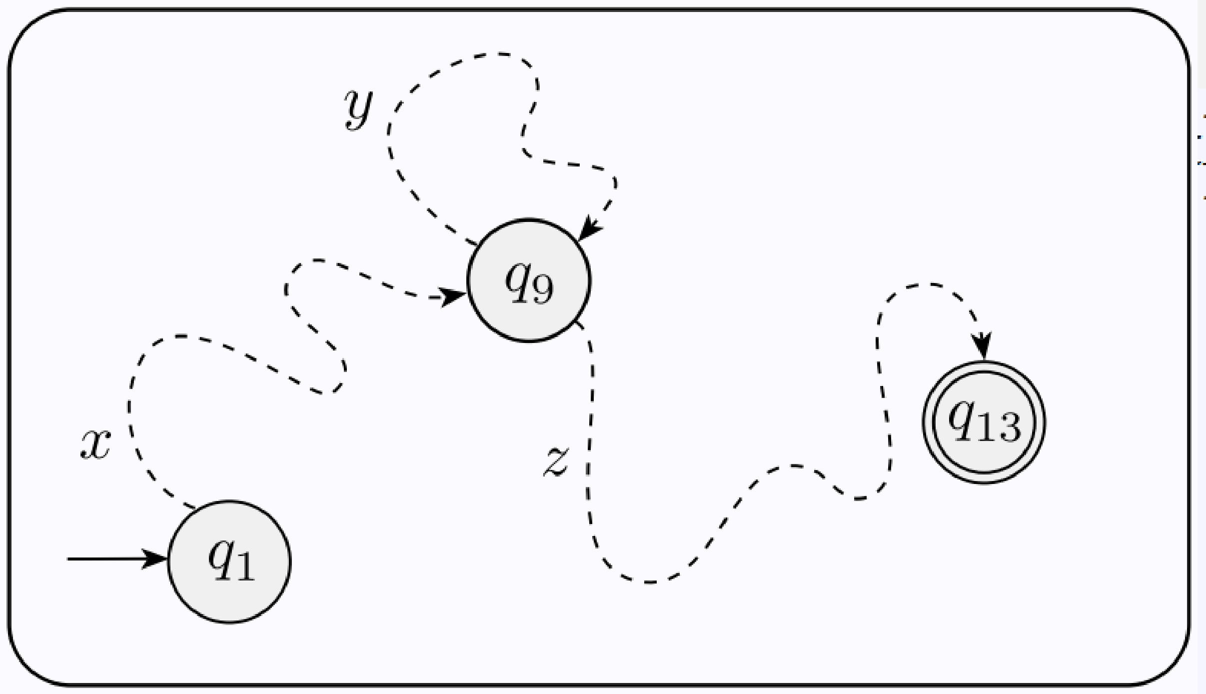

- “Long enough” strings <u>must</u> repeat, since there are only finite states.
  - “Pigeonhole principle”
- Strings that <u>repeat</u> states can be split into:
  - x = the part <u>before</u> any repeating
  - y = the repeated part
  - z = the part <u>after</u> any repeating

Pumping Lemma: Example

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Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

Poll: Conditional Statements

Equivalence of Conditional Statements

- Yes or No? “If X then Y” is equivalent to:
  - “If Y then X” (converse)
    - No
  - “If not X then not Y” (inverse)
    - No
  - “If not Y then not X” (contrapositive)
    - Yes
    - Proof by contradiction relies on this equivalence

Kinds of Mathematical Proof

- Proof by construction
  - Construct the object in question
- Proof by contradiction
  - Proving the contrapositive
- Proof by induction
  - Use to prove properties of recursive definitions or functions

The "Pumping" Lemma

... then the language is **not** regular

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If any of these are **not** true ...

Contrapositive:
"If X then Y" is equivalent to "If **not** Y then **not** X

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Theorem

The language $B = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof.

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5. The contradiction step may require a more detailed case analysis of scenarios. There are three possible cases:

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   5.1 y is all 0s: When pumped, e.g., $xyyz$, the string is not in B because it has more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.

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6. **Alternate Proof:** Last 2 cases not needed; see pumping lemma, condition 3.

Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

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□

Pumping Down

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Check-in Quiz 9/30

On gradescope

End of Class Survey 9/30

See course website