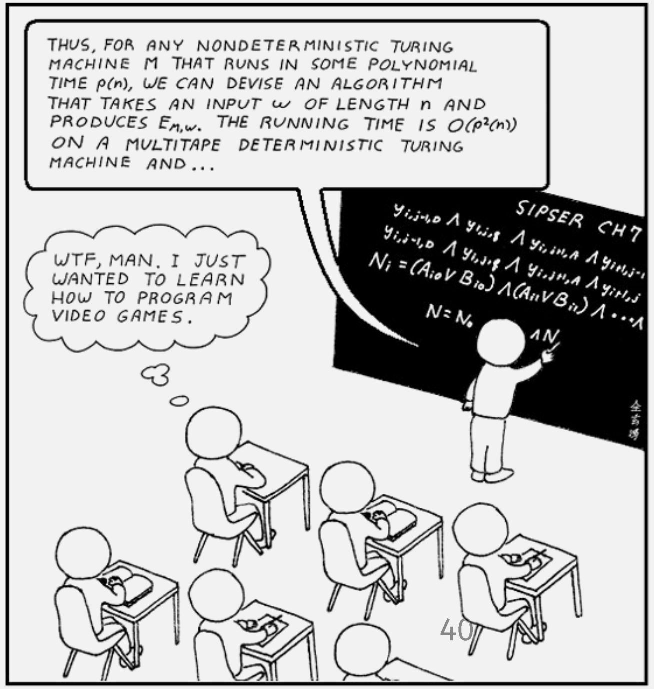
UMB CS420

Nondeterministic TMs

Tuesday, November 1, 2022



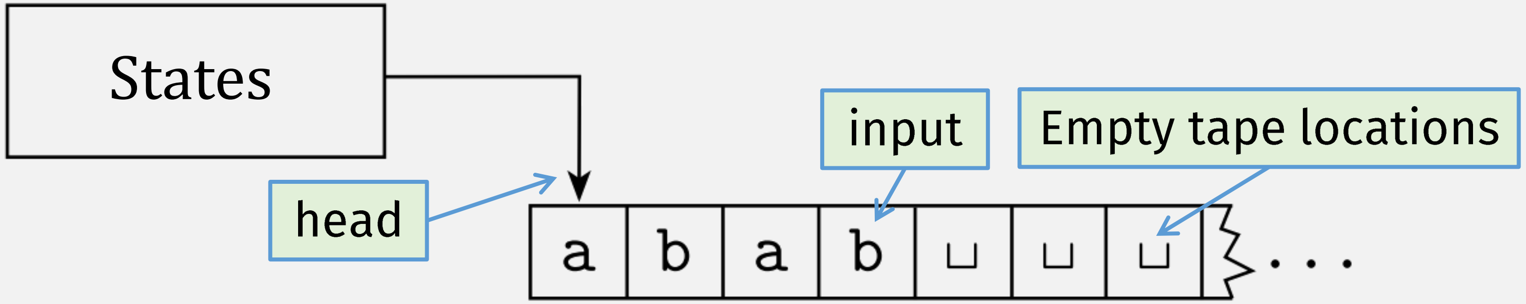
Announcements

- HW 6 in
  - ~~Due Sun 10/30 11:59pm EST~~

- HW 7 out
  - Due Sun 11/6 11:59pm EST

Last Time: Turing Machines

- **Turing Machines** can read and write to arbitrary "tape" cells
  - Tape initially contains input string
- The tape is infinite
  - (to the right)



- On a transition, "head" can move left or right 1 step

Call a language ***Turing-recognizable*** if some Turing machine recognizes it.

Turing Machine: High-Level Description

- M_1 accepts if input is in language $B = \{w\#w \mid w \in \{0,1\}^*\}$

M_1 = "On input string w:

1. Zig-zag across the t… positions on either side of the # symb… the same symbol. … # Cross off symbols as … keep track of which symbols correspond.

2. When all symbols to the … check for any remaining s… symbols remain, *reject*; ot…

We will (mostly) stick to informal descriptions of Turing machines, like this one

(But it must always correspond to some precise formal description)

<u>Analogy</u>:
High-level (e.g., Python) <u>function definitions</u>
vs
<u>assembly language</u>

Turing Machines: Formal Definition

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,

   read write move

5. $q_0 \in Q$ is the start state,
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{reject} \neq q_{accept}$.

Flashback: DFAs vs NFAs

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

vs

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Nondeterministic transition produces set of possible next states

Remember: Turing Machine Formal Definition

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
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Nondeterministic Remember: Turing Machine Formal Definition

A ***Nondeterministic Turing Machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. ~~$\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$~~ → $\delta: Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$
5. $q_0 \in Q$ is the start state,
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Thm: Deterministic TM ⇔ Non-det. TM

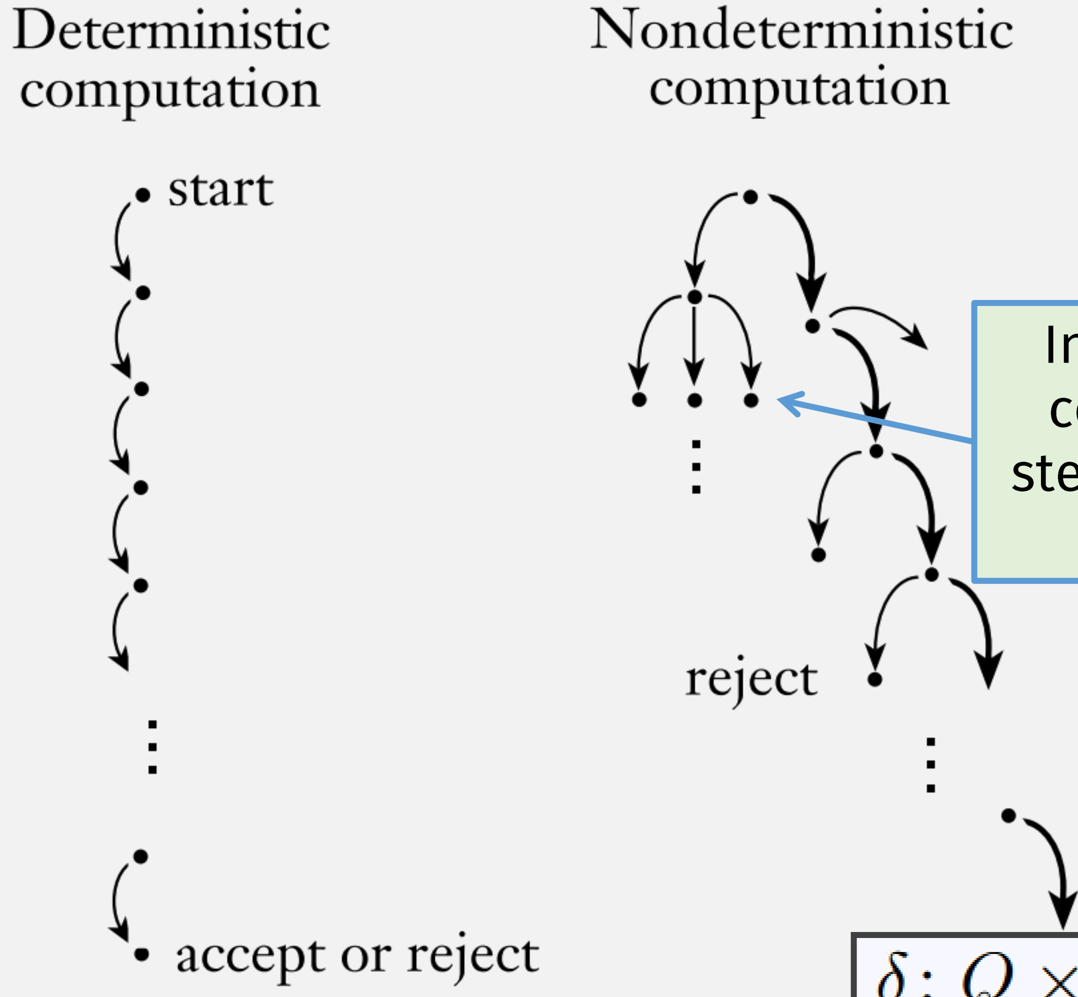
⇒ If a deterministic TM recognizes a language,
then a non-deterministic TM recognizes the language

- Convert: Deterministic TM → Non-deterministic TM …
- … change Deterministic TM δ fn output to a one-element set
  - (just like conversion of DFA to NFA --- HW 2, Problem 2)
- **DONE!**

⇐ If a non-deterministic TM recognizes a language,
then a deterministic TM recognizes the language

- Convert: Non-deterministic TM → Deterministic TM …
- … ???

Review: Nondeterminism



In nondeterministic computation, every step can branch into a set of "states"

What is a "state" for a TM?

$$\delta : Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

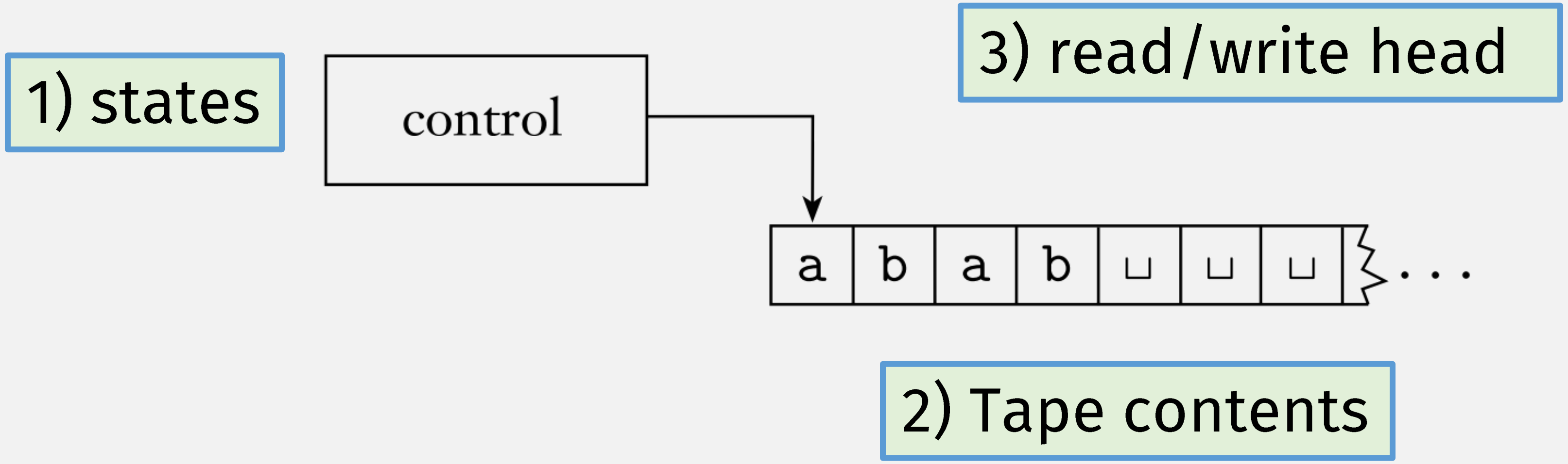
Flashback: PDA Configurations (IDs)

- A **configuration** (or **ID**) is a "snapshot" of a PDA's computation

- 3 components (q, w, γ) :
  - q = the current state
  - w = the remaining input string
  - γ = the stack contents

A **sequence of configurations** represents a **PDA** computation

TM Configuration (ID) = ???



A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
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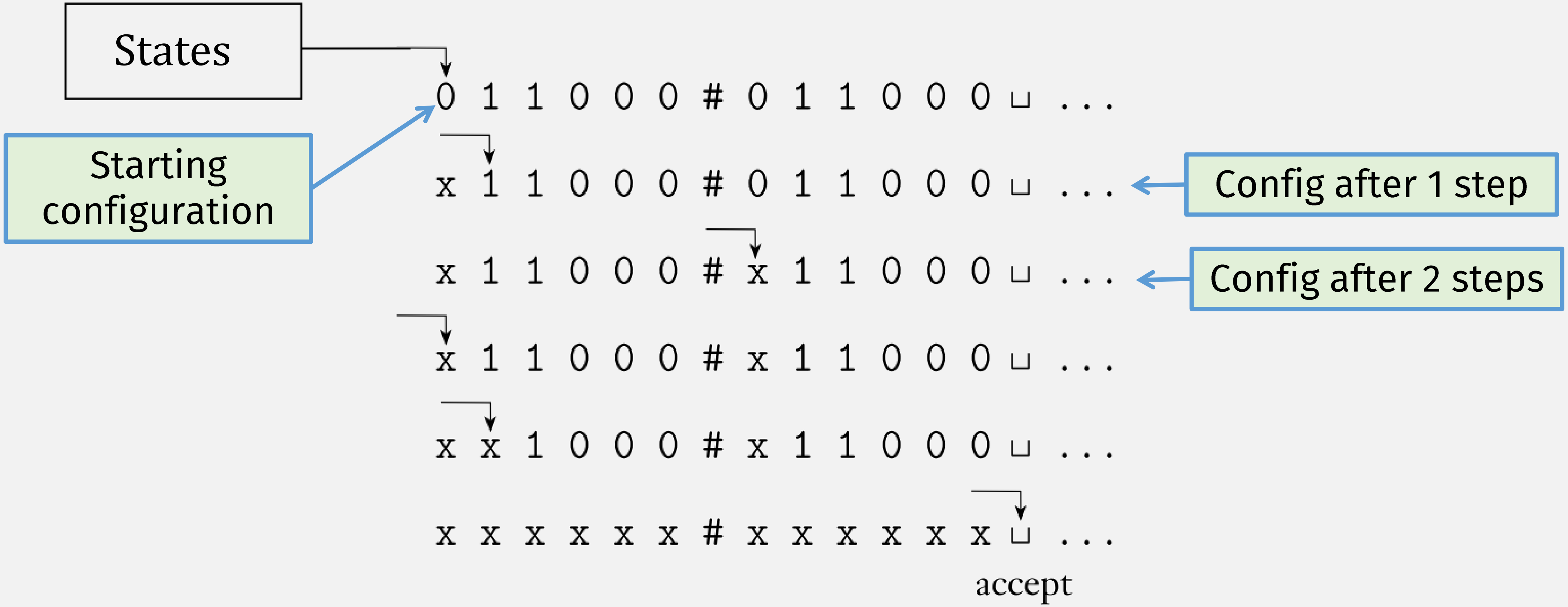
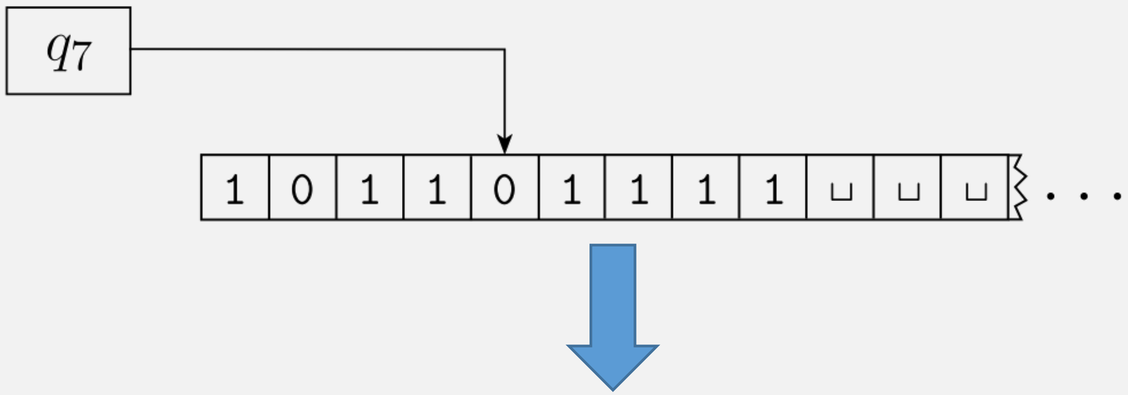
TM Configuration = State + Head + Tape



TM Configuration = State + Head + Tape



$$1011q_7 01111$$

Textual representation of "configuration" (use this in HW)

1st char after state is current head position

TM Computation, Formally

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$$

Single-step

head

Next config

(Right) $\alpha q_1 \mathbf{a} \beta \vdash \alpha \mathbf{x} q_2 \beta$

write

if $q_1, q_2 \in Q$

$\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{R})$

read

$\mathbf{a}, \mathbf{x} \in \Gamma \quad \alpha, \beta \in \Gamma^*$

(Left) $\alpha b q_1 \mathbf{a} \beta \vdash \alpha q_2 b \mathbf{x} \beta$

if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$

Extended

- Base Case

  $I \overset{*}{\vdash} I$ for any ID I

- Recursive Case

  $I \overset{*}{\vdash} J$ if there exists some ID K such that $I \vdash K$ and $K \overset{*}{\vdash} J$

<u>Edge cases</u>:

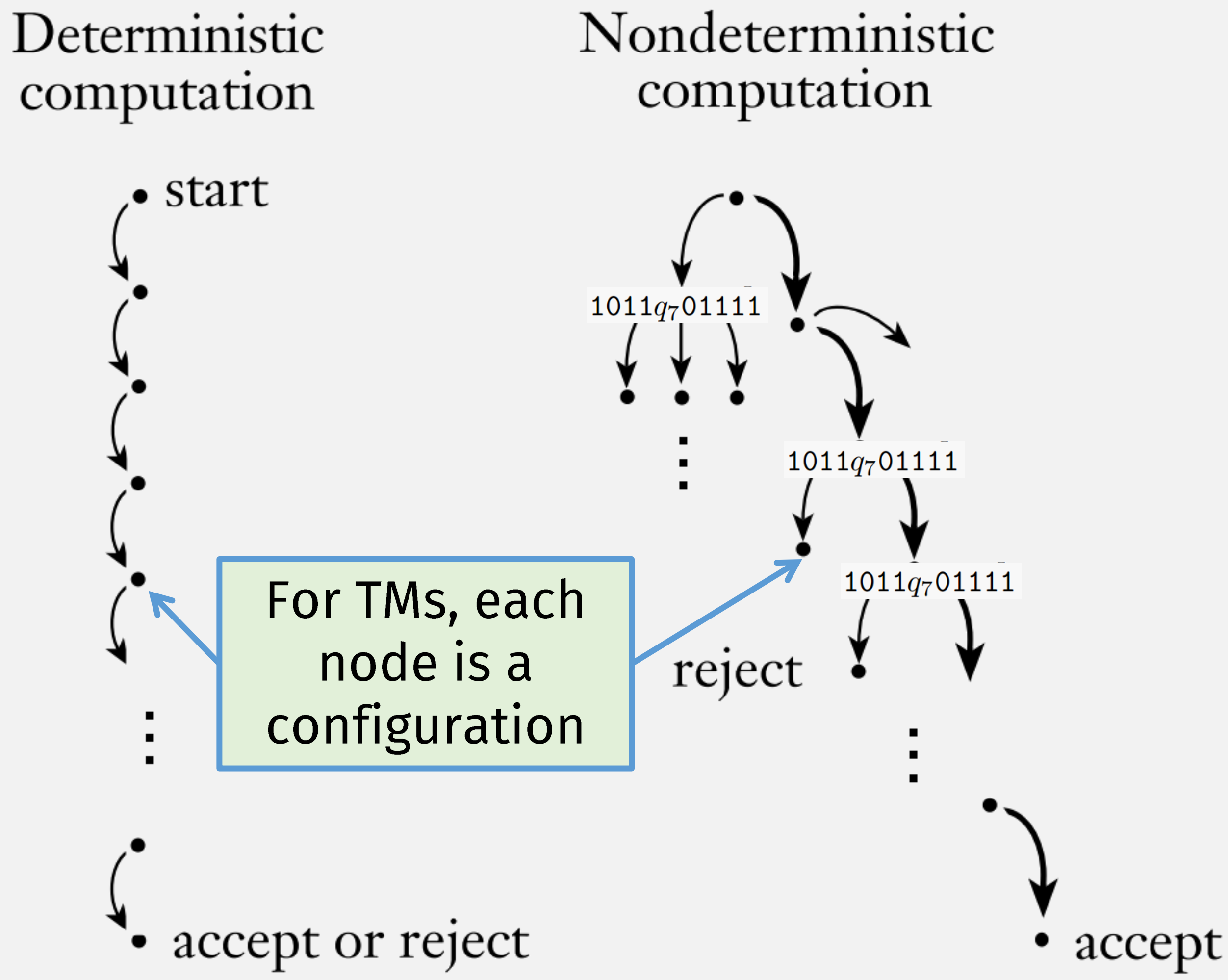
$q_1 \mathbf{a} \beta \vdash q_2 \mathbf{x} \beta$ if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$ (L move, when already at leftmost cell)

Head stays at leftmost cell

$\alpha q_1 \vdash \alpha \llcorner q_2$ if $\delta(q_1, \llcorner) = (q_2, \llcorner, \mathrm{R})$ (R move, when at rightmost filled cell)

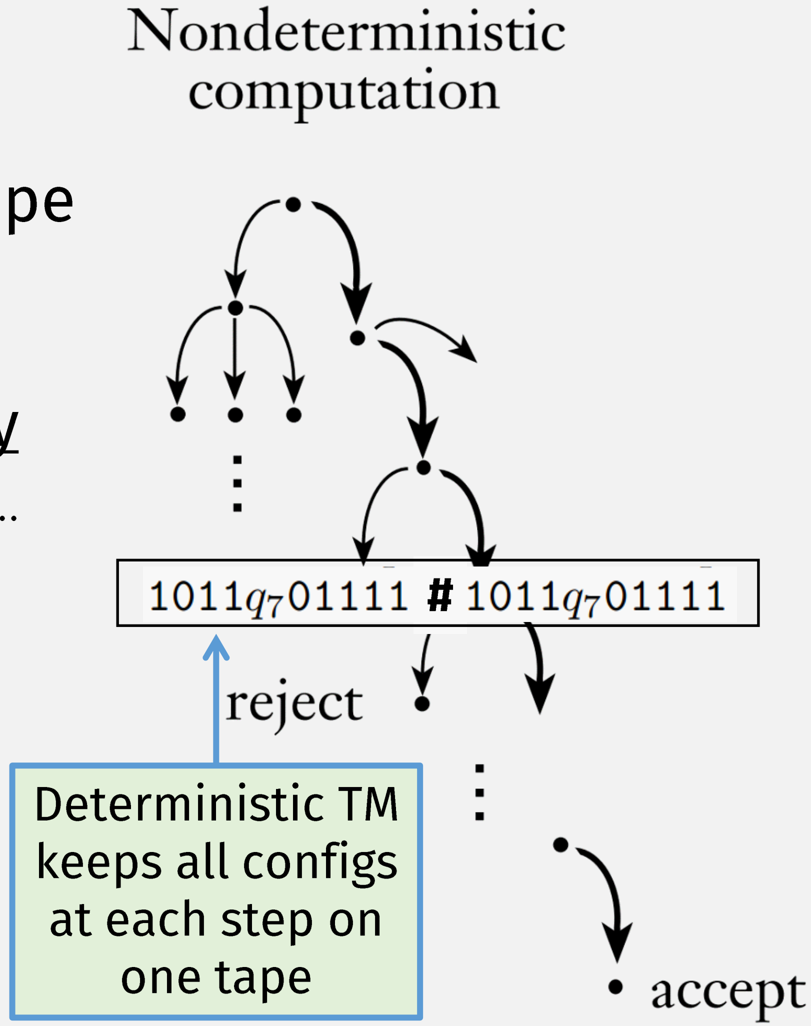
Add blank symbol to config

Nondeterminism in TMs



Nondeterministic TM → Deterministic 1st way

- Simulate NTM with Det. TM:
  - Det. TM keeps multiple configs single tape
    - Like how single-tape TM simulates multi-tape
  - Then run all computations, <u>concurrently</u>
    - I.e., **1 step** on one config, **1 step** on the next, …
  - Accept if any accepting config is found
  - **Important:**
    - Why must we step configs <u>concurrently</u>?



Interlude: Running TMs inside other TMs

If TMs are function definitions, then they can be *called* like functions …

Exercise:

- Given: TMs M_1 and M_2
- Create: TM M that accepts if either M_1 or M_2 accept

“loop” means input string not accepted

M_1	M_2	M
reject	accept	accept
accept	reject	accept

Possible solution #1:

M = on input x,

1. `Call` M_1 with arg x; accept if M_1 accepts
2. `Call` M_2 with arg x; accept if M_2 accepts

Note: This solution would be ok if we knew M_1 and M_2 were **deciders** (which halt on all inputs)

Interlude: Running TMs inside other TMs

If TMs are function definitions, then they can be *called* like functions …

Exercise:

- Given: TMs M_1 and M_2
- Create: TM M that accepts if either M_1 or M_2 accept

… with concurrency!

Possible solution #1:

M = on input x,

1. `Call` M_1 with arg x; accept if M_1 accepts
2. `Call` M_2 with arg x; accept if M_2 accepts

M_1	M_2	M
reject	accept	accept
accept	reject	accept
accept	loops	accept
loops	accept	loops

Possible solution #2:

M = on input x,

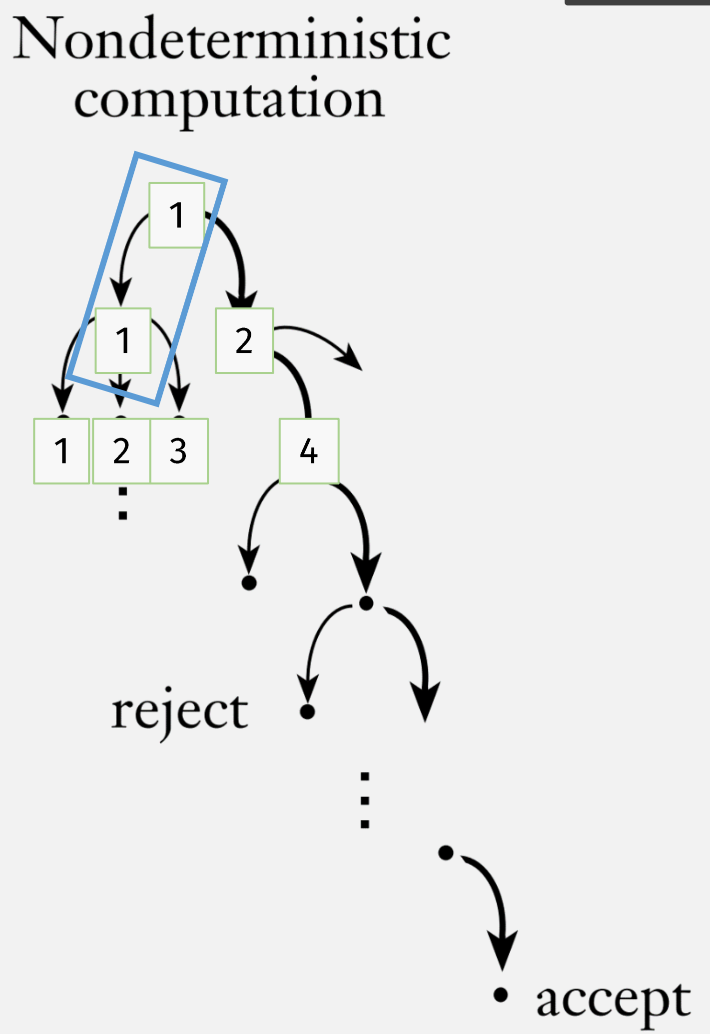
1. `Call` M_1 and M_2 with x concurrently, i.e.,
   a) Run M_1 with x for 1 step; accept if M_1 accepts
   b) Run M_2 with x for 1 step; accept if M_2 accepts
   c) Repeat

M_1	M_2	M
reject	accept	accept
accept	reject	accept
accept	loops	accept
loops	accept	accept

Nondeterministic TM → Deterministic

2nd way
(Sipser)

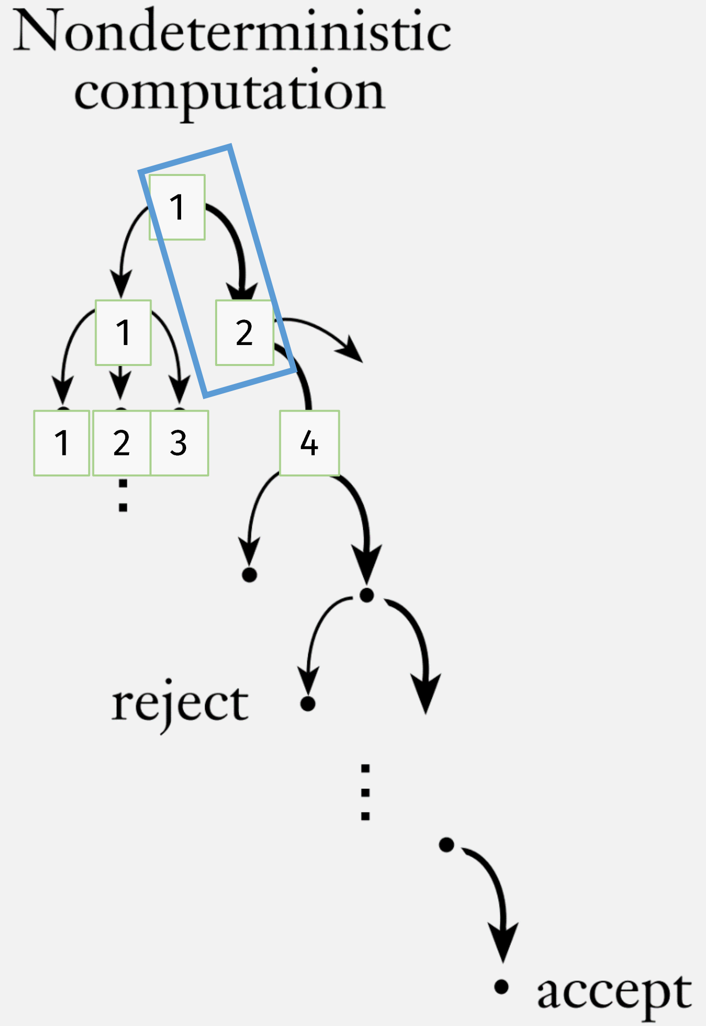
- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - **1-1**



Nondeterministic TM → Deterministic

2nd way
(Sipser)

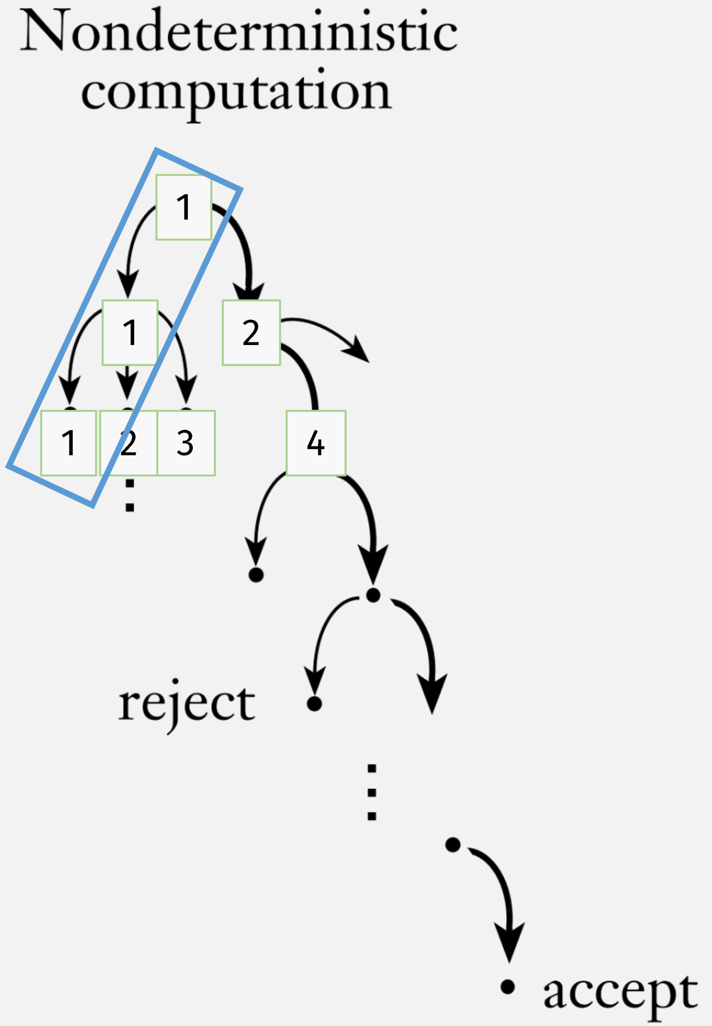
- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - 1-1
    - **1-2**



Nondeterministic TM → Deterministic

2nd way
(Sipser)

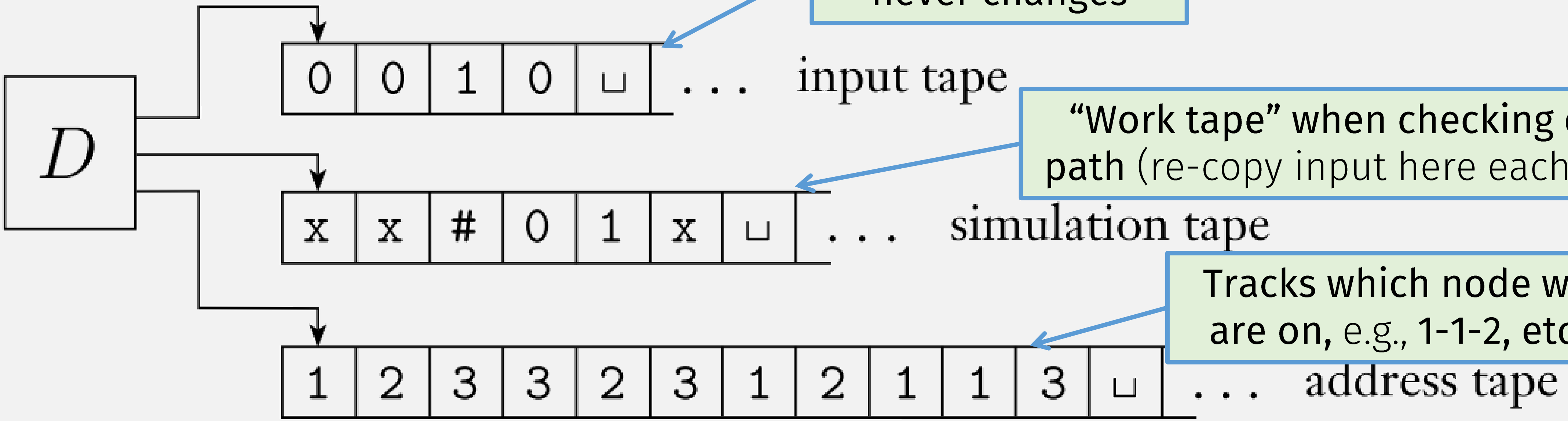
- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - 1-1
    - 1-2
    - **1-1-1**



Nondeterministic TM → Deterministic

2nd way
(Sipser)

Needs 3 tapes



Always has input, never changes

"Work tape" when checking each path (re-copy input here each time)

Tracks which node we are on, e.g., 1-1-2, etc.

Nondeterministic TM ⇔ Deterministic TM

☑ ⇒ If a deterministic TM recognizes a language,
then a nondeterministic TM recognizes the language

- Convert Deterministic TM → Non-deterministic TM

☑ ⇐ If a nondeterministic TM recognizes a language,
then a deterministic TM recognizes the language

- Convert Nondeterministic TM → Deterministic TM

■

Conclusion: These are All Equivalent TMs!

- Single-tape Turing Machine
- Multi-tape Turing Machine
- Non-deterministic Turing Machine

Turing Machines and Algorithms

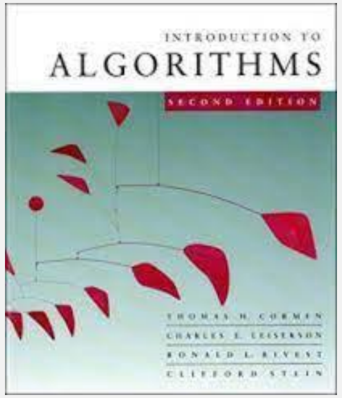
- **Turing Machines** can express any "computation"
  - I.e., a Turing Machine models (`Python`, `Java`) programs (functions)!

- 2 classes of Turing Machines
  - **Recognizers** may loop forever
  - **Deciders** always halt

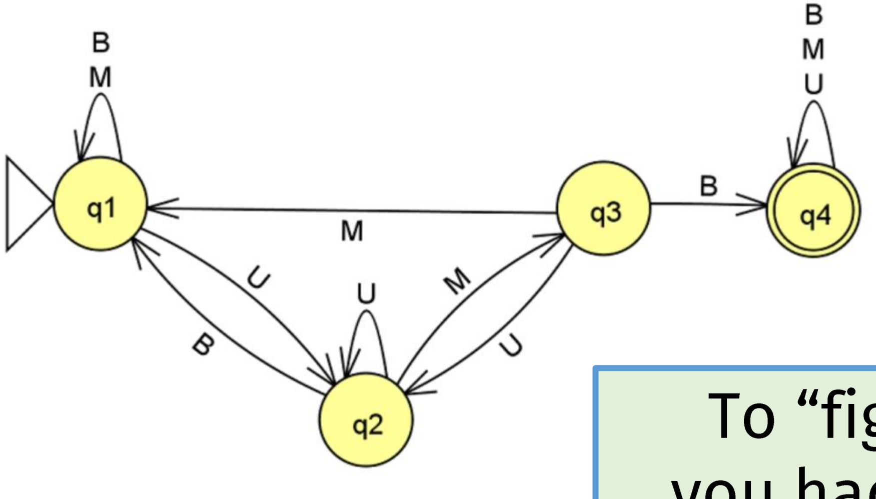
Next

- **Deciders = Algorithms**
  - I.e., an algorithm is any program that always halts

Remember:
TMs = programs



Flashback: HW 1, Problem 1



1. Come up with a formal description for this DFA.

   Recall that a DFA's formal description has five components, $M = (Q, \Sigma, \delta, q_{start}, F)$.

   You may assume that the alphabet contains only the symbols from the diagram.

2. Then for each of the following, say whether the computation represents an accepting computation or not (make sure to review the definition of an accepting computation) If the answer is no, explain why not.:

   a. $\hat{\delta}(q1, \texttt{UUMB})$

   b. $\hat{\delta}(q1, \texttt{UMMB})$

   c. $\hat{\delta}(q2, \texttt{UMBB})$

   d. $\hat{\delta}(q3, \varepsilon)$

   e. $\hat{\delta}(q3, \texttt{UMASSBOSTON})$

This represents computation by a DFA

To "figure out" this computation ...
you had to "do" (meta) computations
(e.g., in your head)

$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

Flashback: DFA Computations

Define the extended transition function: $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$

Base case: $\hat{\delta}(q, \epsilon) = q$

Recursive case: $\hat{\delta}(q, a_1 w_{rest}) = \hat{\delta}(\delta(q, a_1), w_{rest})$

First char

Last chars

Single transition step

Remember: **TMs = programs**

Calculating this computation requires (meta) computation!

Could you implement this (meta) computation as an **algorithm?**

A function: `DFAaccepts(B,w)` returns TRUE if DFA B accepts string w

1) Define "current" state q_{current} = start state q_0
2) For each input char a_i …
   a) Define $q_{\text{next}} = \delta(q_{\text{current}}, a_i)$
   b) Set $q_{\text{current}} = q_{\text{next}}$
3) Return TRUE if q_{current} is an accept state

The language of **`DFAaccepts`**

Function `DFAaccepts(B,w)`
returns TRUE if DFA `B` accepts string `w`

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

But a language is a set of strings?

Interlude: Encoding Things into Strings

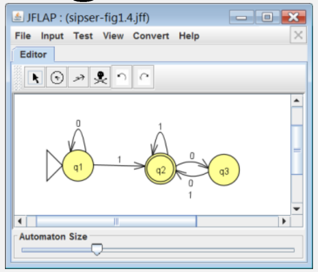
- Definition: **A Turing machine's input is always a string**

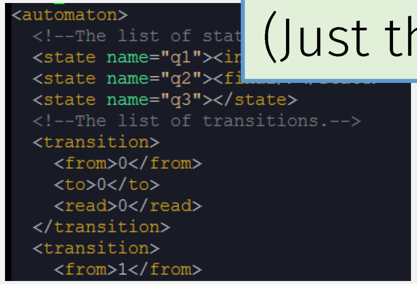
- But: **A TM (program)'s input could also be a list, graph, DFA, …?**
- Solution: **anything used as TM input must be encoded as string**

<u>Notation</u>: **<SOMETHING> = string encoding for SOMETHING**

- **A tuple combines multiple encodings,** e.g., <*B*, *w*> (from prev slide)

<u>Example</u>: **Possible string encoding for a DFA?**





But in this class, <u>we don't care</u> about what the encoding is!
(Just that there is one)

$$(Q, \Sigma, \delta, q_0, F)$$

<u>(written as string)</u>

Interlude: High-Level TMs and Encodings

A high-level TM description:

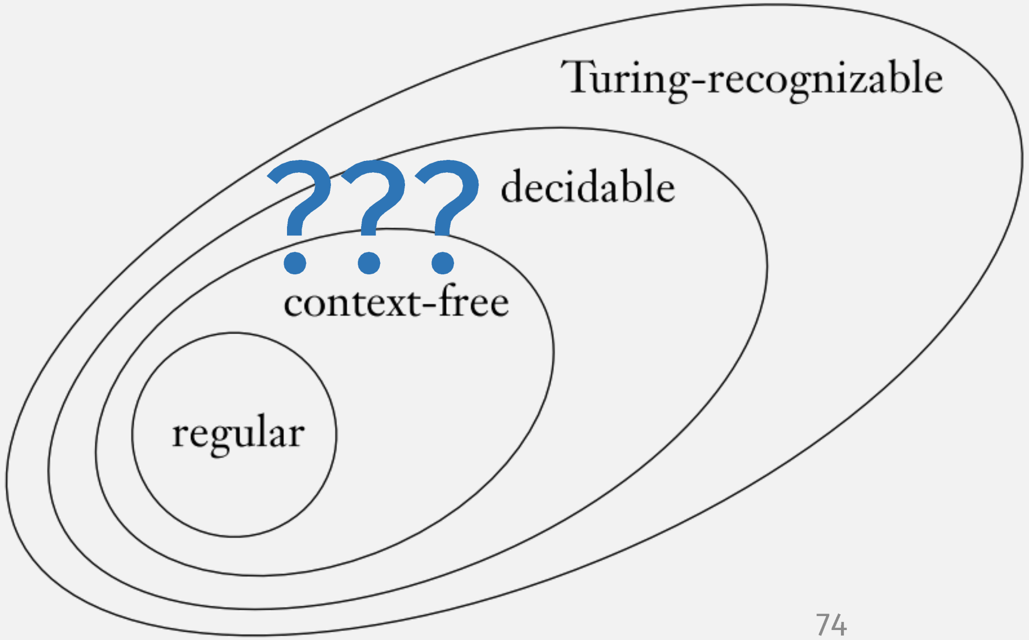
1. Doesn’t need to describe exactly how input string is encoded
2. Assumes input is a “valid” encoding
   - Invalid encodings are implicitly rejected

The language of **`DFAaccepts`**

$$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

- **`DFAaccepts`** is a Turing machine
- But **is it a decider** or **recognizer**?
  - I.e., **is it an algorithm**?
- **To show it's an algo, need to <u>prove</u>:**

$$A_{\mathsf{DFA}} \text{ is a decidable language}$$



How to prove that a language is decidable?

- Create a Turing machine that **decides** that language!

Remember:

- A **decider** is Turing Machine that always halts
  - I.e., for any input, it either accepts or rejects it.
  - It must never go into an infinite loop

How to Design Deciders

- If TMs = Programs …

  … then **Creating** a TM = Programm**ing**

- E.g., if HW asks “Show that lang L is decidable” …
  - .. you must create a TM that decides L; to do this …
  - … think of how to write a (halting) program that does what you want

Next Time: A_{DFA} is a decidable language

$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider for A_{DFA} :

Check-in Quiz 11/1

On gradescope