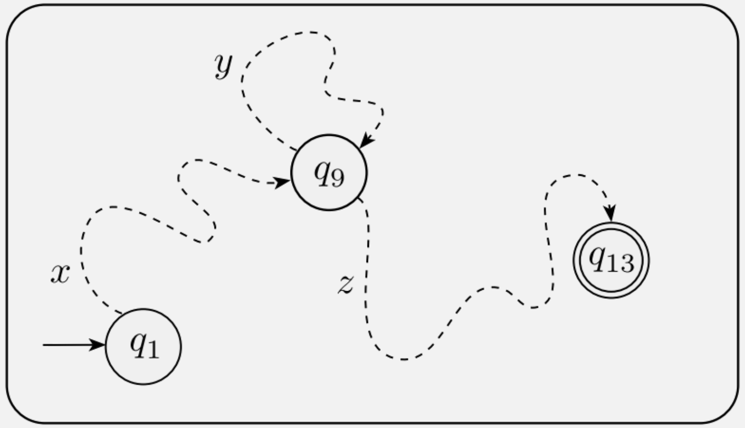
Examples with the Pumping Lemma

Wed Feb 24, 2021



Logistics

- HW3 solutions posted (soon)
- HW4 due Sunday 2/28 11:59pm EST
- Questions?

Last time: The Pumping Lemma says:

For all strings in a regular language that are "long enough" (i.e., length p) ...

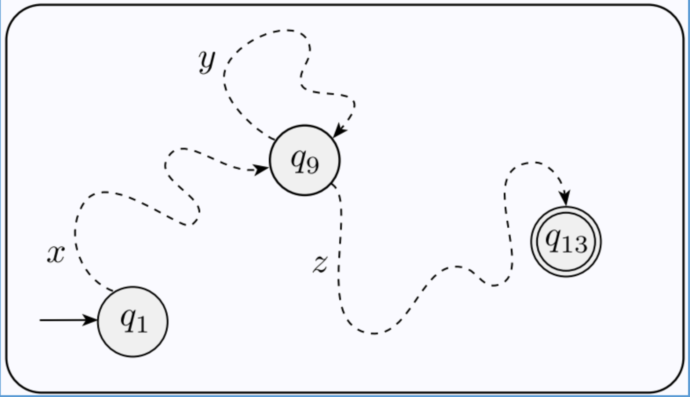
Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

... these strings must be divisible into three pieces (call them x, y, and z) ...

... where repeating the middle piece y results in a "pumped" string is also in the language

Also, repeating part:
- can't be empty string
- must be in the first p characters



tl;dr:
Long enough strings means repeated states

Last time: Equivalence of Contrapositive

- “If X then Y” is equivalent to … ?
  - “If Y then X” (converse)
    - No!
  - “If not X then not Y” (inverse)
    - No!
  - ✓“If not Y then not X” (contrapositive)
    - **Yes**!
    - Proof by contradiction uses this equivalence

The Pumping Lemma is an If-Then Stmt

... then the language is **not** regular

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Just need one counterexample!

Contrapositive: If (**any** of) these are **not** true ...

IMPORTANT NOTE:
The pumping lemma **cannot** be used to show that a language is regular, only that is is non-regular

Pumping Lemma: Non-Regularity Example

Let B be the language $\{0^n 1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Theorem

The language $B = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.

Theorem

The language $B = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that B is a regular language.
   Then it must satisfy the pumping lemma where p is the pumping length.

Theorem

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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that B is a regular language.
   Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string $0^p 1^p$.

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4. Show contradiction of assumption: Because $s \in B$ and has length $> p$, the pumping lemma guarantees that s can be split into three pieces $s = xyz$ where $xy^i z \in B$ for $i \geq 0$. But we will show this is impossible ...

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5. The contradiction step typically requires detailed case analysis of scenarios.
   There are three possible cases:

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5. The contradiction step typically requires detailed case analysis of scenarios.
   There are three possible cases:
   5.1 y is all 0s: Pumped strings, e.g., $xyyz$, are not in B because they have more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.

Theorem

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   5.2 y is all 1s: Same as above.

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   5.2 y is all 1s: Same as above.
   5.3 y has both 0s and 1s: Pumped strings preserve equal counts, but is out of order and therefore not in B, breaking condition 1.

Theorem

The language $B = \{0^n 1^n \mid n \geq 0\}$ is not regular.

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   5.2 y is all 1s: Same as above.
   5.3 y has both 0s and 1s: Pumped strings preserve equal counts, but is out of order and therefore not in B, breaking condition 1.
6. Conclusion: Since all cases result in contradiction, B must not be regular.

Theorem

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Proof.

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1. State the kind of proof: The proof is by contradiction.
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   - 5.1 y is all 0s: Pumped strings, e.g., $xyyz$, are not in B because they have more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.
   - 5.2 y is all 1s: Same as above.
   - 5.3 y has both 0s and 1s: Pumped strings preserve equal counts, but is out of order and therefore not in B, breaking condition 1.
6. **Alternate Proof:** Last 2 cases not needed; see pumping lemma, condition 3.

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

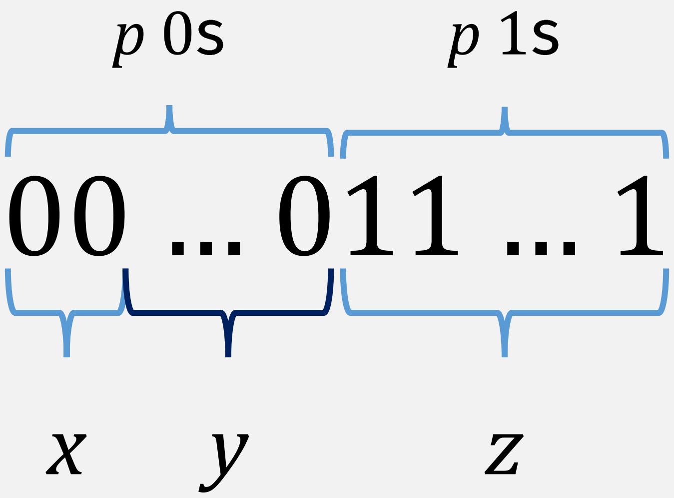
1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Possible Split: y = all 0s

- Assumption: 0^n1^n is a regular language (must satisfy pumpi
- Counterexample = 0^p1^p

But pumping lemma requires **only one** pumpable splitting

- If xyz chosen so y contains
  - all 0s



So we must show that **every splitting** produces a contradiction

- Pumping y: produces a string with more 0s than 1s
  - This string is not in the language 0^n1^n
  - This means that 0^n1^n does not satisfy the pumping lemma
  - Which means that that 0^n1^n is a not regular lang
  - This is a **contradiction** of the assumption!

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
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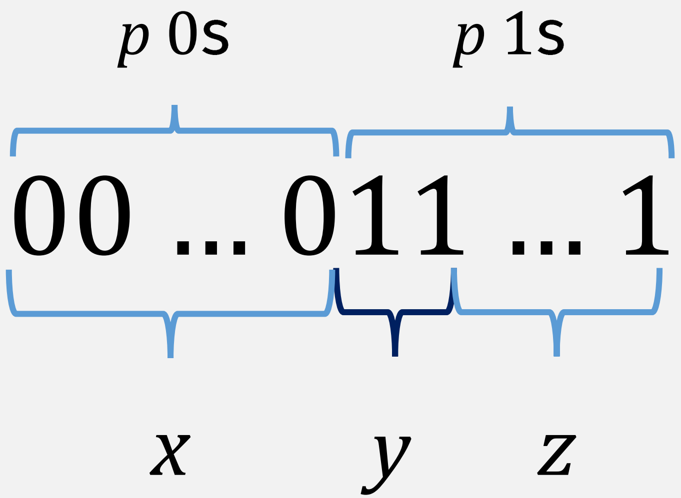
Possible Split: y = all 1s

- Assumption: 0^n1^n is a regular language (must satisfy pumping lemma)
- Counterexample = 0^p1^p



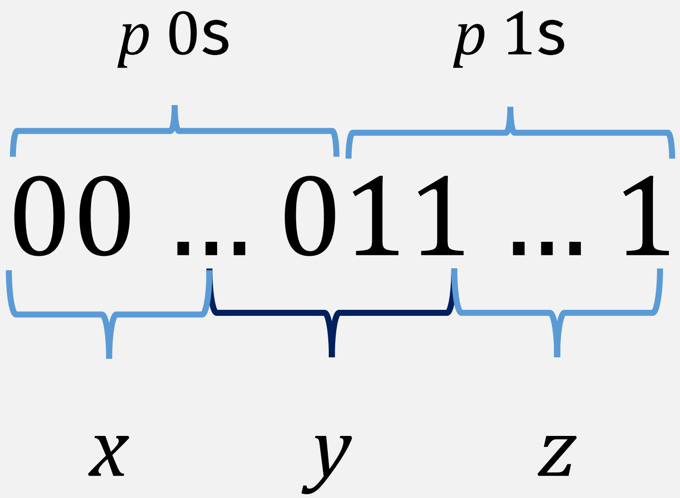
- If xyz chosen so y contains
  - all 1s
- Is this string pumpable?
  - No!
  - By the same reasoning as in the previous slide

Possible Split: y = 0s and 1s

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- Assumption: 0^n1^n is a regular language (must satisfy pumping lemma)
- Counterexample = 0^p1^p



- If xyz chosen so y contains
  - both 0s and 1s
- Is this string pumpable?
  - No!
  - Pumped string will have equal 0s and 1s
  - But they will be in the wrong order: so there is still a **contradiction**!

Did we examine every possible splitting?

Yes. But maybe we don't have to.

Last time: The Pumping Lemma says:

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Also, repeating part y:
- can't be empty string
- must be in the first p characters

p 0s

00 … 011 … 1

y must be in here!

Pumping Lemma: How to use Condition 3

Let $F = \{ww|\ w \in \{0,1\}^*\}$. We show that F is nonregular

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.

Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that F is regular. Then it must satisfy the pumping lemma where p is the pumping length.

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Theorem
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Proof.
This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that F is regular. Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string 0^p10^p1.

Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that F is regular. Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string 0^p10^p1.
4. Show contradiction of assumption: Because $s \in F$ and has length $> p$, the pumping lemma guarantees that s can be split into three pieces $s = xyz$ where $xy^iz \in F$ for $i \geq 0$. But we will show this is impossible ...

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Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Proof.

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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that F is regular. Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string $0^p 1 0^p 1$.
4. Show contradiction of assumption: Because $s \in F$ and has length $> p$, the pumping lemma guarantees that s can be split into three pieces $s = xyz$ where $xy^i z \in F$ for $i \geq 0$. But we will show this is impossible ...
5. This time there is only one possible case, but we must explain why. According to condition 3 of the pumping lemma $|xy| \leq p$. So y is all 0s. But then $xyyz \notin F$, breaking condition 1 of the pumping lemma. So we have a contradiction.

Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ *is not regular.*

Proof.

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5. This time there is only one possible case, but we must explain why. According to condition 3 of the pumping lemma $|xy| \leq p$. So y is all 0s. But then $xyyz \notin F$, breaking condition 1 of the pumping lemma. So we have a contradiction.
6. Conclusion: Since all cases result in contradiction, F must not be regular.

Pumping Lemma: Pumping Down

use the pumping lemma to show that $E = \{0^i 1^j | \ i > j\}$ is not regular.

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Pumping Down

Theorem

The language $E = \{0^i 1^j \mid i > j\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.

Pumping Down

Theorem

The language $E = \{0^i 1^j \mid i > j\}$ *is not regular.*

Proof.

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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that E is regular. Then it must satisfy the pumping lemma where p is the pumping length.

Pumping Down

Theorem

The language $E = \{0^i 1^j \mid i > j\}$ *is not regular.*

Proof.

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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that E is regular. Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string $0^{p+1}1^p$.

Pumping Down

Theorem

The language $E = \{0^i 1^j \mid i > j\}$ *is not regular.*

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that E is regular. Then it must satisfy the pumping lemma where p is the pumping length.
3. Present counterexample: Choose s to be the string $0^{p+1}1^p$.
4. Show contradiction of assumption: Because $s \in E$ and has length $> p$, the pumping lemma guarantees that s can be split into three pieces $s = xyz$ where $xy^iz \in E$ for $i \geq 0$. But we will show this is impossible ...

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5. Again, one possible case. According to condition 3 of the pumping lemma $|xy| \leq p$. So y is all 0s. But then $xz \notin E$ ($i = 0$), breaking condition 1 of the pumping lemma. So we have a contradiction.

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6. Conclusion: Since all cases result in contradiction, E must not be regular.

□

Check-in Quiz 2/24

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