NP

Monday, April 26, 2021

Announcements

- HW10 due Tues 4/27 11:59pm EST
- HW11 out soon
  - Due Tues 5/4 11:59pm EST
- <u>Reminder</u>: Submitted HW must be in your own words
  - Not “your own words”: Submitting answers from the internet
  - Not “your own words”: Changing variables / rearranging sentences
  - Suggestion: Looking into “clean room” design

<u>Last Time</u>: Polynomial Time (**P**)

Mathematicians are weird.

DEFINITION 7.12

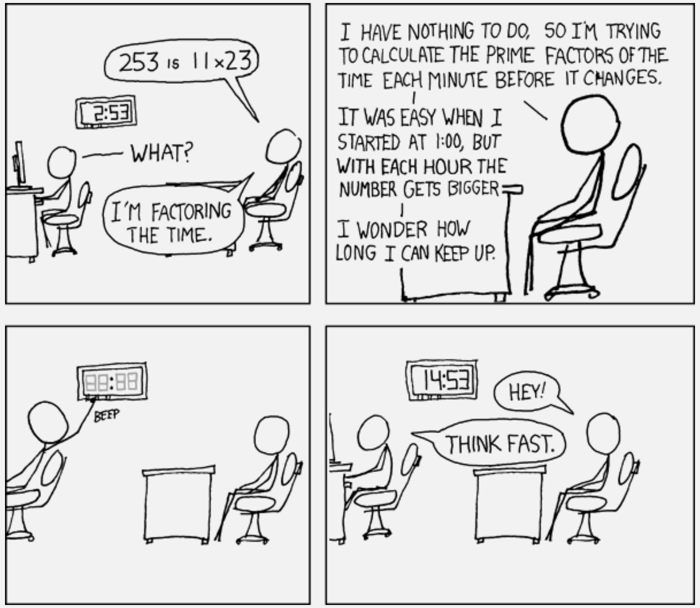
P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

- Roughly corresponds to **solvable** vs **unsolvable** problems:
  - Problems in **P** = "solvable"
  - Problems outside **P** = "unsolvable"

Today: Search vs Verification

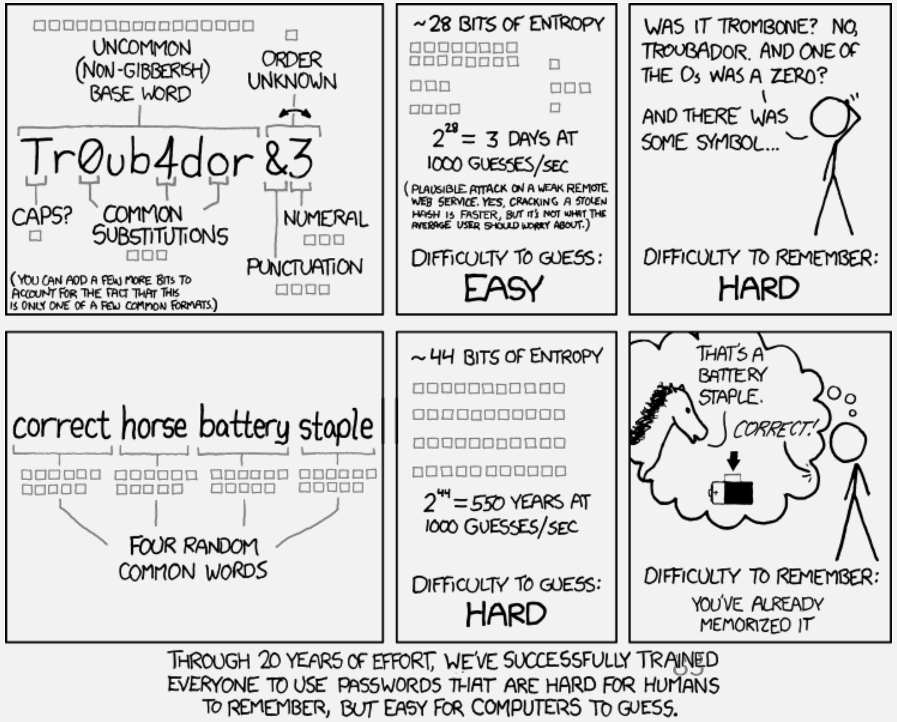
- **Search** problems are often **unsolvable**
- But, **verification** of search results is usually **solvable**



EXAMPLES

- Factoring
  - **Unsolvable**: Find factors of 8633
  - **Solvable**: Verify 89 and 97 are factors of 8633

- Passwords
  - **Unsolvable**: Find my `umb.edu` password
  - **Solvable** : Verify whether my `umb.edu` password is …
    - “`correct horse battery staple`”



Last Time: The *PATH* Problem

$PATH = \{\langle G, s, t \rangle | \ G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- The **search** problem:
  - Exponential time (brute force) algorithm (n^n):
    - Check all possible paths and see if any connects s and t
  - Polynomial time algorithm:
    - Do a breadth-first search (roughly), marking "seen" nodes as we go

PROOF A polynomial time algorithm M for *PATH* operates as follows.

M = "On input $\langle G, s, t \rangle$, where G is a directed graph with nodes s and t:

1. Place a mark on node s.
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of G. If an edge (a, b) is found going from a marked node a to an unmarked node b, mark node b.
4. If t is marked, *accept*. Otherwise, *reject*."

Verifying a *PATH*

$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

- The **verification** problem:
  - Given some path p in G, check that it is a path from s to t
  - Let $\boldsymbol{m}$ = longest possible path = # edges in G
  - Verifier V = On input <G, s, t, p>, where p is some set of edges:

NOTE: extra argument p

1. Check some edge in p has "from" node s; mark and set it as "current" edge
   - Max steps = $O(\boldsymbol{m})$
2. Loop: While there remains unmarked edges in p:
   a) Find the "next" edge in p, whose "from" node is the "to" node of "current" edge
   b) If found, then mark that edge and set it as "current", else reject
   - Each loop: Max steps $O(\boldsymbol{m})$
   - # loops: at most $\boldsymbol{m}$ times
   - Total looping time = $O(\boldsymbol{m}^2)$
3. Check "current" edge has "to" node t; if yes accept, else reject

- Total time = $O(\boldsymbol{m}) + O(\boldsymbol{m}^2) = O(\boldsymbol{m}^2)$ = polynomial in m

PATH can be verified in polynomial time

Verifiers, Formally

$PATH = \{\langle G, s, t \rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

DEFINITION 7.18

A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c \rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

extra argument:
can be any string that helps to find a result in poly time (is often just a result itself)

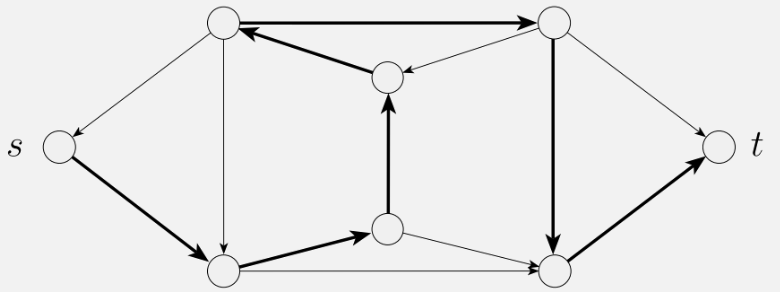
certificate, or ***proof***

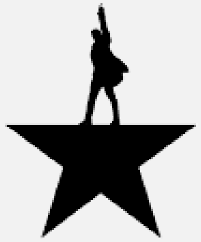
- NOTE: a cert c must be at most length n^k, where n = length of w
  - Why?
- So *PATH* is polynomially verifiable

The *HAMPATH* Problem

$$HAMPATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph with a Hamiltonian path from } s \text{ to } t\}$$

- A Hamiltonian path goes through every node in the graph





- The **Search** problem:
  - Exponential time (brute force) algorithm:
    - Check all possible paths and see if any connect s and t using all nodes
  - Polynomial time algorithm:
    - We don't know if there is one!!!
- The **Verification** problem:
  - Still $O(\boldsymbol{m^2})$!
  - *HAMPATH* is polynomially verifiable, but <u>not</u> polynomially decidable

The class **NP**

DEFINITION 7.19

NP is the class of languages that have polynomial time verifiers.

- *PATH* is in **NP**, and **P**
- *HAMPATH* is in **NP**, but <u>not</u> **P**

NP = Nondeterministic polynomial time

DEFINITION 7.19

NP is the class of languages that have polynomial time verifiers.

THEOREM 7.20

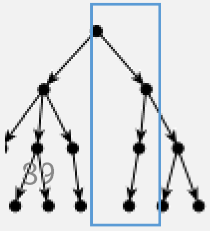
A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

=> If a lang L is in **NP**, then we know it has a poly time verifier V

- Need to: Create NTM deciding L: on input w =
  - Nondeterministically run V with w and all possible certificates c

<= If L has NTM decider N,

- Need to: show L is in **NP**, ie it has polytime verifier V: on input <w, c> =
  - Convert N to deterministic TM, and run it on w, but take only one computation path
  - Let certificate c dictate which computation path to follow



P vs NP

DEFINITION 7.7

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, **TIME($t(n)$)**, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

DEFINITION 7.12

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

DEFINITION 7.21

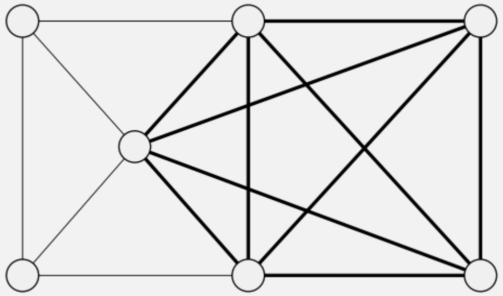
$$\textbf{NTIME}(\boldsymbol{t(n)}) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time nondeterministic Turing machine}\}.$$

COROLLARY 7.22

$$\mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k).$$

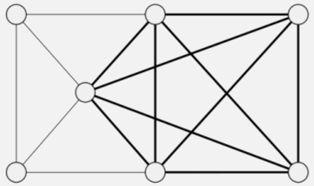
More **NP** Problems

- $CLIQUE = \{\langle G, k\rangle |\ G \text{ is an undirected graph with a } k\text{-clique}\}$
  - A clique is a subgraph where every two nodes are connected
  - A *k*-clique contains *k* nodes



- $$SUBSET\text{-}SUM = \{\langle S, t\rangle |\ S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$$

Theorem: *CLIQUE* is in NP



$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

PROOF IDEA The clique is the certificate.

PROOF The following is a verifier V for *CLIQUE*.

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a subgraph with k nodes in G. $O(\boldsymbol{k})$
2. Test whether G contains all edges connecting nodes in c. $O(\boldsymbol{k}^2)$
3. If both pass, *accept*; otherwise, *reject*."

DEFINITION 7.18

A ***verifier*** for a language A is an algorithm V, where

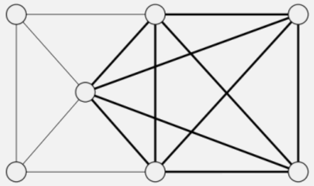
$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

DEFINITION 7.19

NP is the class of languages that have polynomial time verifiers.

Proof 2: *CLIQUE* is in NP



CLIQUE $= \{\langle G, k\rangle \mid G$ is an undirected graph with a k-clique$\}$

N = "On input $\langle G, k\rangle$, where G is a graph:

1. Nondeterministically select a subset c of k nodes of G.
2. Test whether G contains all edges connecting nodes in c.
3. If yes, *accept*; otherwise, *reject*."

"try all subgraphs"

$O(\boldsymbol{k}^2)$

To prove a lang L is in **NP**, create either a:

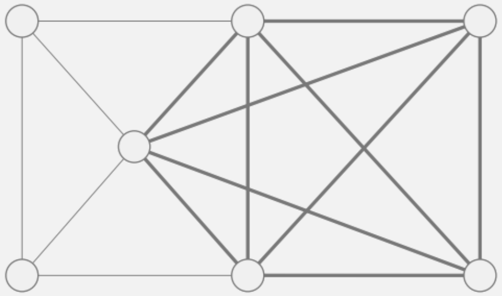
- **Deterministic** poly time **verifier**
- **Nondeterministic** poly time **decider**

THEOREM 7.20

A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

More **NP** Problems

- $CLIQUE = \{\langle G, k\rangle |\ G$ is an undirected graph with a k-clique$\}$
  - A clique is a subgraph where every two nodes are connected
  - A k-clique contains k nodes



- $SUBSET\text{-}SUM = \{\langle S, t\rangle |\ S = \{x_1, \ldots, x_k\}$, and for some $\{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}$, we have $\Sigma y_i = t\}$
  - Some subset of a set of numbers S must sum to some total t
  - e.g., $\langle\{4, 11, 16, 21, 27\}, 25\rangle \in SUBSET\text{-}SUM$

Theorem: *SUBSET-SUM* is in NP

$$SUBSET\text{-}SUM = \{\langle S, t\rangle \mid S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$$

PROOF IDEA The subset is the certificate.

To prove a lang is in **NP**, create <u>either</u>:
- **Deterministic** poly time **verifier**
- **Nondeterministic** poly time **decider**

PROOF The following is a verifier V for *SUBSET-SUM*.

V = "On input $\langle\langle S, t\rangle, c\rangle$:
1. Test whether c is a collection of numbers that sum to t.
2. Test whether S contains all the numbers in c.
3. If both pass, *accept*; otherwise, *reject*."

ALTERNATIVE PROOF We can also prove this theorem by giving a nondeterministic polynomial time Turing machine for *SUBSET-SUM* as follows.

N = "On input $\langle S, t\rangle$:
1. Nondeterministically select a subset c of the numbers in S.
2. Test whether c is a collection of numbers that sum to t.
3. If the test passes, *accept*; otherwise, *reject*."

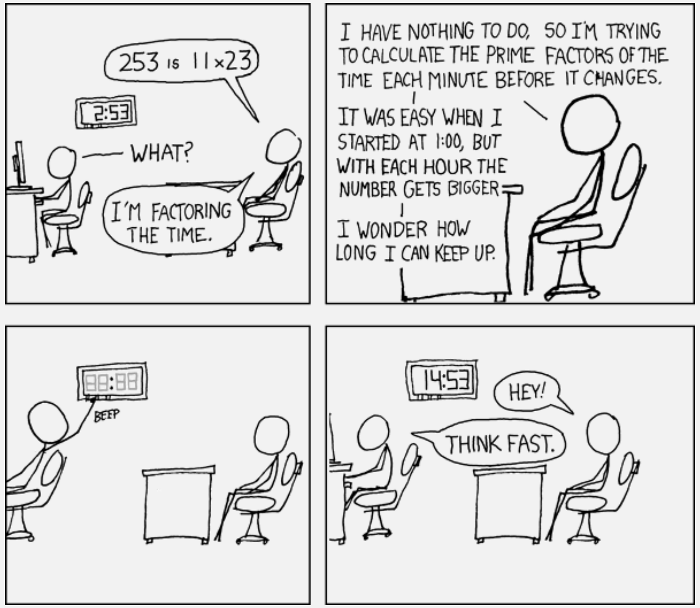
$$COMPOSITES = \{x \mid x = pq,\ \text{for integers } p, q > 1\}$$

- A composite number is <u>not</u> prime

- *COMPOSITES* is polynomially verifiable
  - i.e., it's in **NP**
  - i.e., factorability is in **NP**

- A certificate could be:
  - Some factor that is not 1

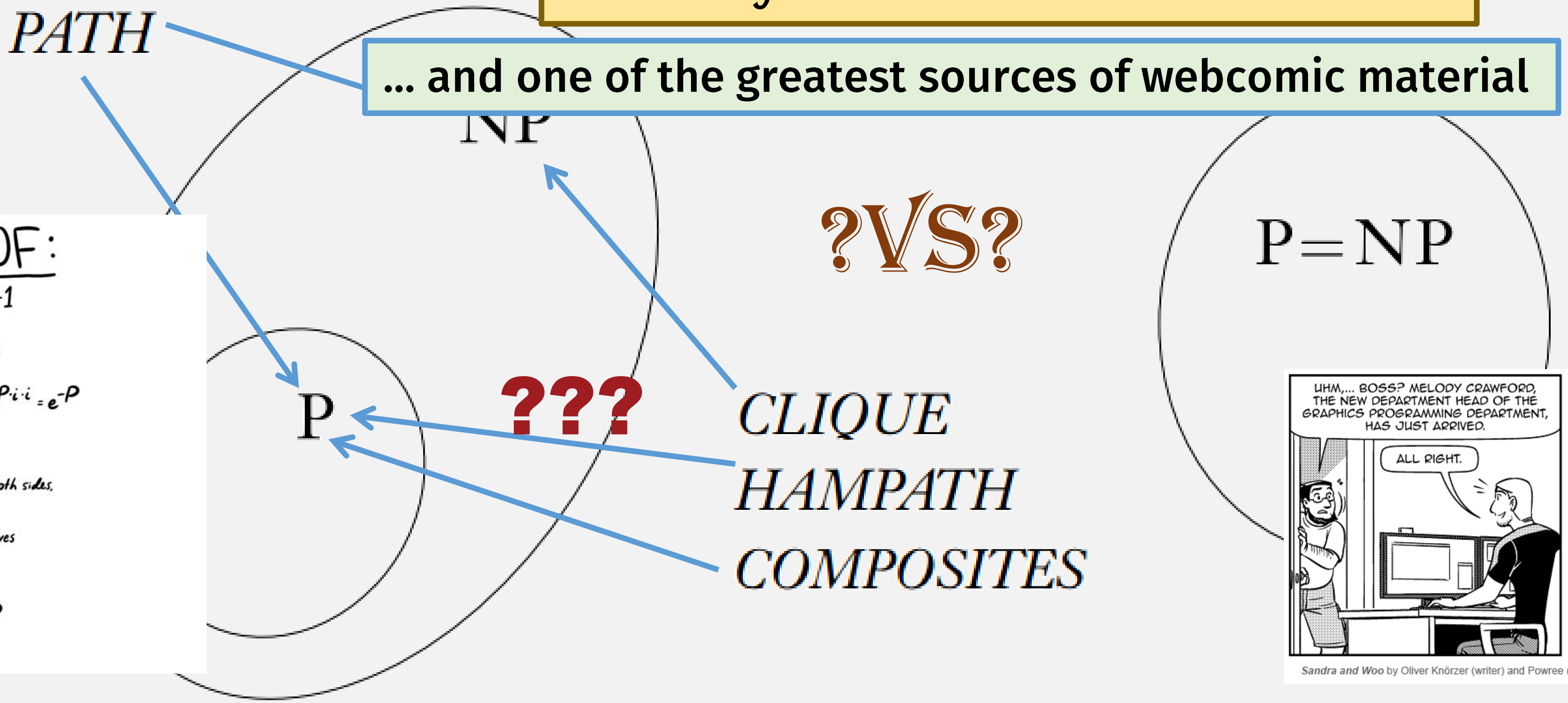
- Checking existence of factors (or not, i.e., testing primality) …
  - … is also poly time
  - But only discovered recently (2002)

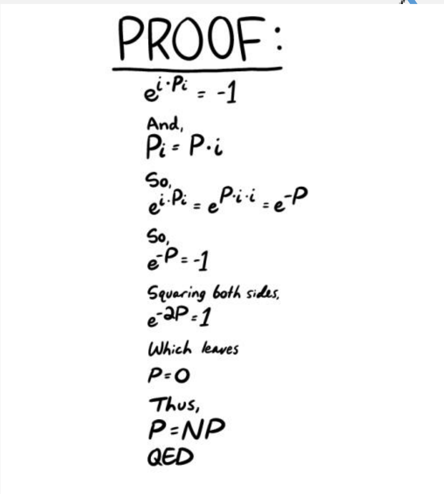


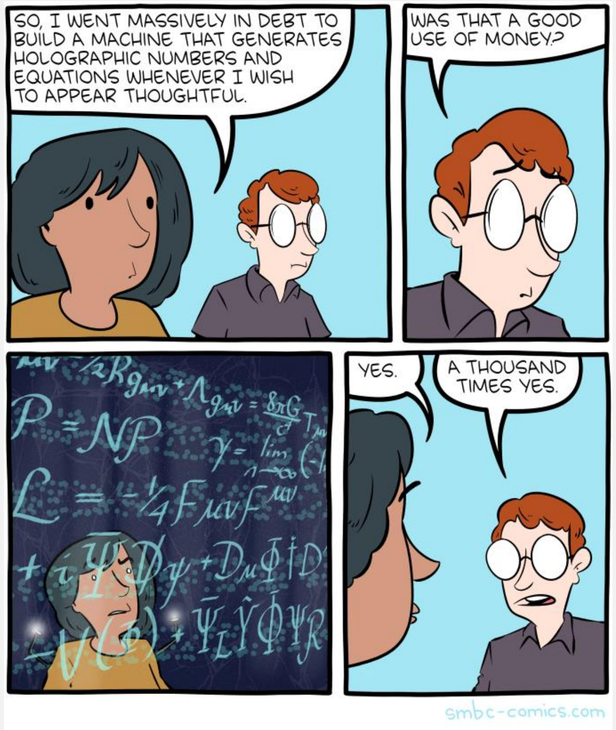
Question: Does P = NP?

One of the greatest unsolved mysteries in science

... and one of the greatest sources of webcomic material







Sandra and Woo by Oliver Knörzer (writer) and Powree (artist) – www.sandraandwoo.com

How do you prove an algorithm <u>doesn't</u> have a poly time algorithm?
(in general it's hard to prove that something <u>doesn't</u> exist)

Implications if **P = NP**

- Every problem with a "brute force" solution also has an efficient solution
- I.e., "unsolvable" problems are "solvable"
- <u>BAD</u>:
  - Cryptography needs unsolvable problems
  - Near perfect AI learning, recognition
- <u>GOOD</u>: Optimization problems are solved
  - Overcrowding or world hunger solved?
  - Abundant energy resources?

Who doesn't like niche NP jokes?

Progress on whether **P = NP** ?

- Some, but still not close

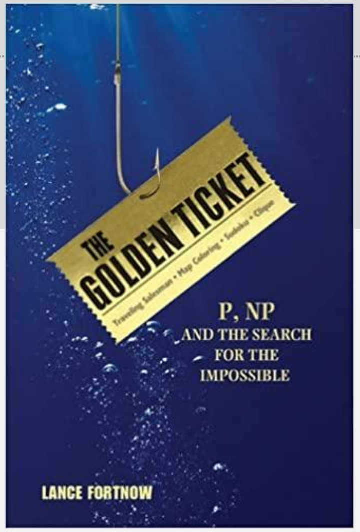
$$P \stackrel{?}{=} NP$$

Scott Aaronson*

The Status of the P Versus NP Problem

By Lance Fortnow

Communications of the ACM, September 2009, Vol. 52 No. 9, Pages 78-86
10.1145/1562164.1562186



- One important concept discovered:
  - **NP**-Completeness (next time)

Next time: **NP**-Completeness

Must look at langs in general, can't just look at any single lang

DEFINITION 7.34

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and — easy
2. every A in NP is polynomial time reducible to B. — hard????

- How does this help the **P** = **NP** problem?

THEOREM 7.35

If B is NP-complete and $B \in \text{P}$, then $\text{P} = \text{NP}$

Check-in Quiz 4/26

On gradescope