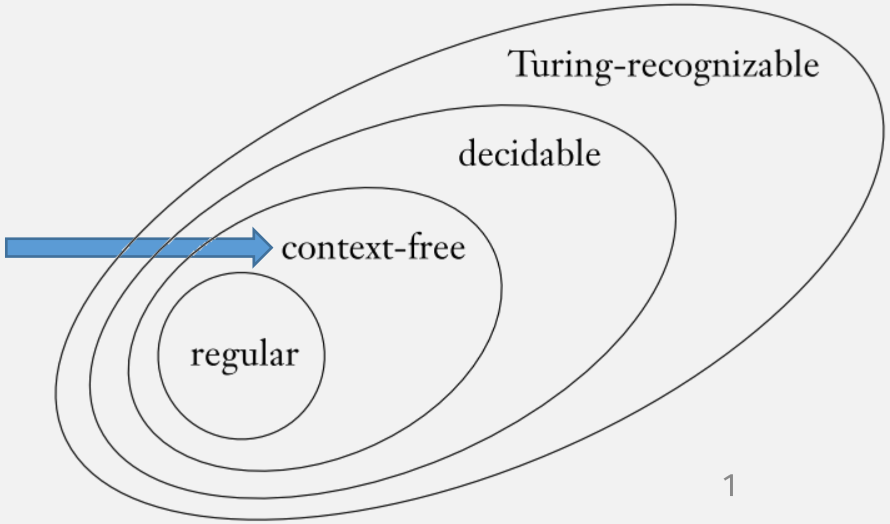
UMB CS 420

Context-Free Languages (CFLs)

Wednesday, February 16, 2022



Announcements

- HW3, Problem 3 (rev strings) has a new requirement!
  - See piazza post and hw3 page

- HW3 due Sun 2/20 11:59pm EST

- Reminder: No class next Monday 2/21

Last Time:

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

- **Assume** the language is regular
- So it must follow the Pumping Lemma:
  - All strings longer than length p ...
  - ... must be splitable into xyz ... where y is "pumpable"
- Find **counterexample** where Pumping Lemma does not hold: $\boxed{0^p1^p}$
- Therefore, the language is not regular
  - This is the contrapositive of the Pumping Lemma
  - And also a **contradiction** of the assumption!

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If this language is not regular, then what is it???

Maybe? … a **context-free language** (CFL)?

A Context-Free Grammar (CFG)

terminals

Top variable is
Start variable

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Variables
(a.k.a., **nonterminals**)

Substitution rules
(a.k.a., **productions**)

terminals (analogous to a DFA’s alphabet)

A Context-Free Grammar (CFG)

grammar G_1

R is

Top variable is **Start variable**

terminals

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Variables (a.k.a., **nonterminals**)

Substitution rules (a.k.a., **productions**)

terminals (analogous to a DFA's alphabet)

A **CFG describes a context-free language!**

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$V = \{A, B\}$,

$\Sigma = \{0, 1, \#\}$,

$S = A$,

Analogies

Regular Language	Context-Free Language (CFL)
Regular Expression	Context-Free Grammar (CFG)
A Reg expr describes a Regular lang	A CFG describes a CFL

CFG Practical Application:
Used to describe **programming language syntax!**

Java Syntax: Described with CFGs

ORACLE

Java SE > Java SE Specifications > Java Language Specification

Chapter 2. Grammars

Prev

Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification to define the lexical and syntactic structure of a progran

2.1. Context-Free Grammars

A *context-free grammar* consists of a number of *productions*. Each production has an abstract symbol called a *nonterminal* as its *lef hand side*, and a sequence of one or more nonterminal and *terminal* symbols as its *right-hand side*. For each grammar, the terminal symbols are drawn from a specified *alphabet*.

Starting from a sentence consisting of a single distinguished nonterminal, called the *goal symbol*, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

2.2. The Lexical Grammar

A *lexical grammar* for the Java programming language is given in §3. This grammar has as its terminal symbols the characters of th Unicode character set. It defines a set of productions, starting from the goal symbol *Input* (§3.5), that describe how sequences of Unicode characters (§3.1) are translated into a sequence of input elements (§3.5)

https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html

(partially) Python Syntax: Described with a CFG

10. Full Grammar specification

This is the full Python grammar, as it is read by the parser generator and used to parse Python source files:

```
# Grammar for Python

# NOTE WELL: You should also follow all the steps listed at
# https://devguide.python.org/grammar/

# Start symbols for the grammar:
#       single_input is a single interactive statement;
#       file_input is a module or sequence of commands read from an input file;
#       eval_input is the input for the eval() functions.
#       func_type_input is a PEP 484 Python 2 function type comment
# NB: compound_stmt in single_input is followed by extra NEWLINE!
# NB: due to the way TYPE_COMMENT is tokenized it will always be followed by a NEWLINE
single_input: NEWLINE | simple_stmt | compound_stmt NEWLINE
file_input: (NEWLINE | stmt)* ENDMARKER
eval_input: testlist NEWLINE* ENDMARKER
```

(indentation checking probably not describable with a CFG)

https://docs.python.org/3/reference/grammar.html

Many Other Language (partially)

~~Python~~ Syntax: Described with a CFG

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Generating Strings with a CFG

$$G_1 =$$

1st rule

$$\begin{aligned} A &\to 0A1 \\ A &\to B \\ B &\to \# \end{aligned}$$

A CFG represents a context free language!

Strings in CFG's language
= all possible generated strings

$$\boxed{L(G_1) \text{ is } \{0^n\#1^n \mid n \geq 0\}}$$

Stop when string is all terminals

A CFG **generates** a string, by repeatedly applying substitution rules:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Start variable | After applying 1st rule | 1st rule again | 1st rule again | Use 2nd rule | Use last rule

Derivations: Formally

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Let $G = (V, \Sigma, R, S)$

Single-step

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

Where:

$\alpha, \beta \in (V \cup \Sigma)^*$ ← Strings of terminals and variables

$A \in V$ ← Variable

$A \to \gamma \in R$ ← Rule

Extended Derivation

Base case: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \alpha$ (0 steps)

Recursive case: (multistep)

- If $\alpha \underset{G}{\Rightarrow} \beta$ and $\beta \overset{*}{\underset{G}{\Rightarrow}} \gamma$

  Single step

  Recursive call

- Then: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \gamma$

Formal Definition of a CFL

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

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3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$$G = (V, \Sigma, R, S)$$

$$L(G) = \left\{ w \in \Sigma^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w \right\}$$

Any language that can be generated by some context-free grammar is called a ***context-free language***

Flashback: $\{0^n 1^n | \, n \geq 0\}$

- Pumping Lemma says it's not a regular language
- It's a context-free language!
  - Proof?
  - Come up with CFG describing it …
  - <u>Hint</u>: It's similar to:

$$A \to 0A1$$
$$A \to B$$
$$B \to \cancel{\#}\ \varepsilon$$

$L(G_1)$ is $\{0^n \cancel{\#} 1^n | \, n \geq 0\}$

A String Can Have Multiple Derivations

$$\begin{aligned} \langle \text{EXPR} \rangle &\to \langle \text{EXPR} \rangle + \langle \text{TERM} \rangle \mid \langle \text{TERM} \rangle \\ \langle \text{TERM} \rangle &\to \langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle \mid \langle \text{FACTOR} \rangle \\ \langle \text{FACTOR} \rangle &\to (\langle \text{EXPR} \rangle) \mid \mathtt{a} \end{aligned}$$

Want to generate this string: **a + a × a**

- $\underline{\text{EXPR}} \Rightarrow$
- $\text{EXPR} + \underline{\text{TERM}} \Rightarrow$
- $\text{EXPR} + \text{TERM} \times \underline{\text{FACTOR}} \Rightarrow$
- $\text{EXPR} + \text{TERM} \times \mathbf{a} \Rightarrow$

  ...

RIGHTMOST DERIVATION

- $\underline{\text{EXPR}} \Rightarrow$
- $\underline{\text{EXPR}} + \text{TERM} \Rightarrow$
- $\underline{\text{TERM}} + \text{TERM} \Rightarrow$
- $\underline{\text{FACTOR}} + \text{TERM} \Rightarrow$
- $\mathbf{a} + \text{TERM}$

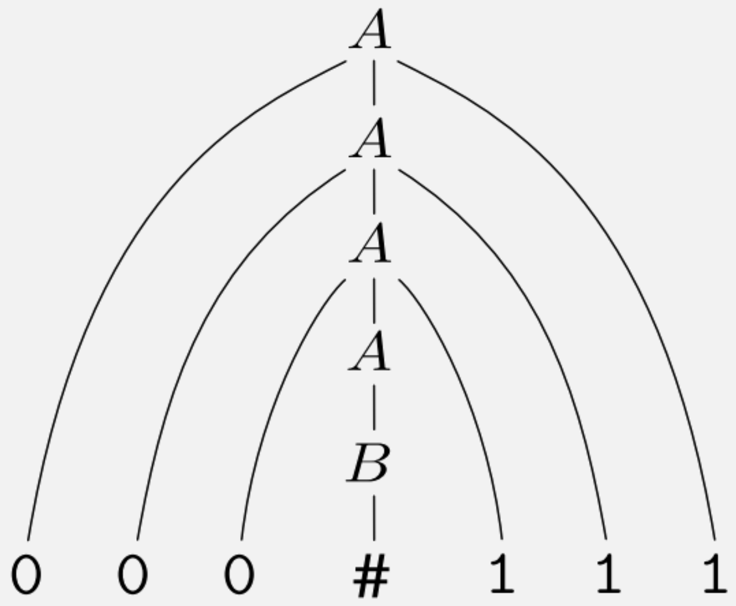
  ...

LEFTMOST DERIVATION

Derivations and Parse Trees

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

A derivation may also be represented as a **parse tree**



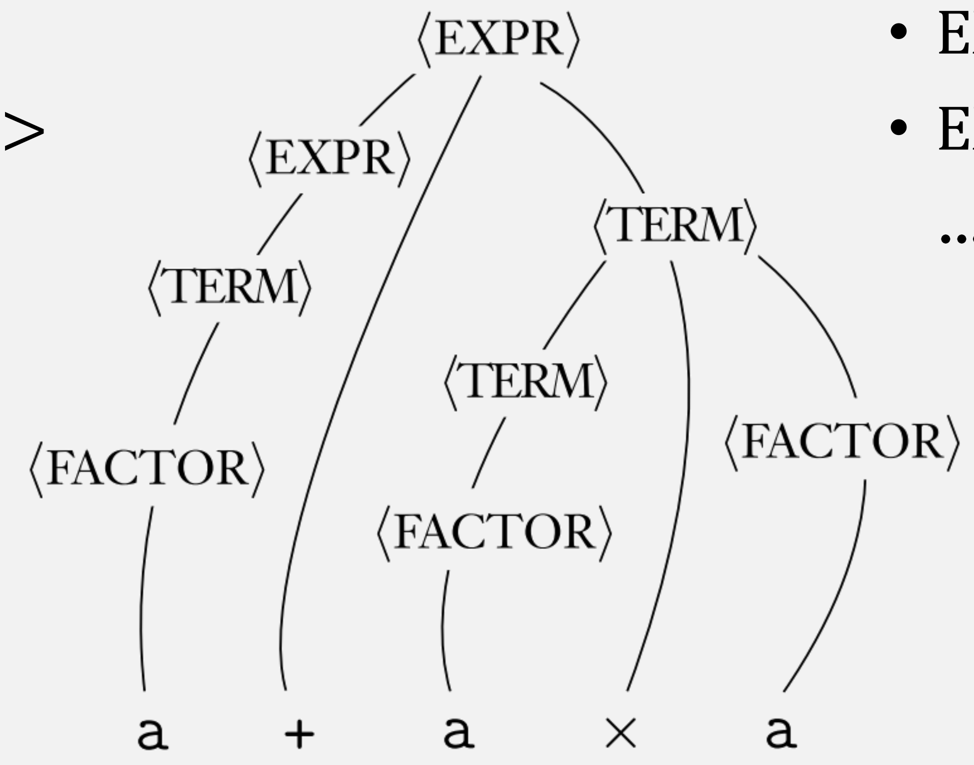
Multiple Derivations, Single Parse Tree

Leftmost deriviation

- EXPR =>
- EXPR + TERM =>
- TERM + TERM =>
- FACTOR + TERM =>
- a + TERM

  …

Same parse tree



Rightmost deriviation

- EXPR =>
- EXPR + TERM =>
- EXPR + TERM x FACTOR =>
- EXPR + TERM x a=>

…

Since the "meaning" (i.e., parse tree) is same, by convention we just use **leftmost** derivation

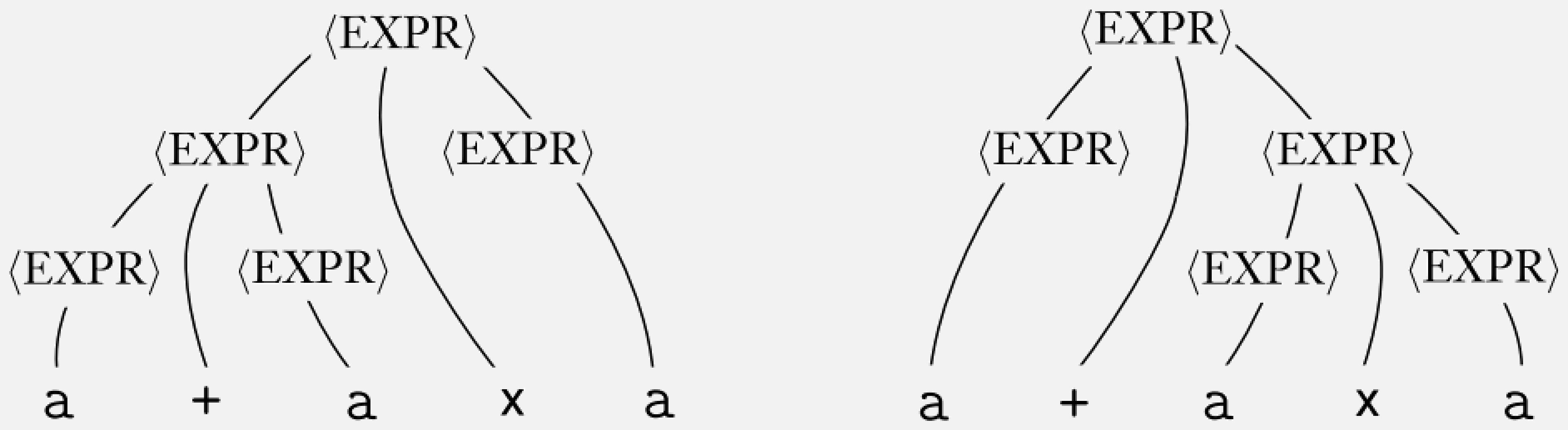
A Parse Tree gives "meaning" to a string

Ambiguity

grammar G_5:

$$\langle \mathrm{EXPR} \rangle \to \langle \mathrm{EXPR} \rangle \texttt{+} \langle \mathrm{EXPR} \rangle \mid \langle \mathrm{EXPR} \rangle \texttt{x} \langle \mathrm{EXPR} \rangle \mid \texttt{(} \langle \mathrm{EXPR} \rangle \texttt{)} \mid \texttt{a}$$

Same **string,**
different **derivation,**
<u>and</u> different **parse tree!**



Ambiguity

A string w is derived ***ambiguously*** in context-free grammar G if it has two or more different leftmost derivations. Grammar G is ***ambiguous*** if it generates some string ambiguously.

An ambiguous grammar can give a string <u>multiple meanings</u>!
(why is this **bad**?)

Real-life Ambiguity ("Dangling" **`else`**)

- What is the result of this C program?

```
if (1) if (0) printf("a"); else printf("2");
```

```
if (1)
  if (0)
    printf("a");
  else
    printf("2");
```

VS

```
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

This string has <u>2 parsings</u>, and thus <u>2 meanings</u>!

Ambiguous grammars are confusing. In a (programming) language, a string (program) should have only **one meaning** (result).

Problem is, there's no guaranteed way to create an unambiguous grammar (up to language designers to "be careful")

Designing Grammars : Basics

1. Think about what you want to “link” together

- E.g., 0^n1^n
  - $A \rightarrow 0A1$
  - # 0s and # 1s are “linked”

- E.g., XML
  - $\text{ELEMENT} \rightarrow \text{<TAG>CONTENT</TAG>}$
  - Start and end tags are “linked”

2. Start with small grammars and then combine (just like FSMs)

Designing Grammars: Building Up

- Start with small grammars and then combine (just like FSMs)
  - To create a grammar for the language $\{0^n1^n | n \geq 0\} \cup \{1^n0^n | n \geq 0\}$
  - First create grammar for lang $\{0^n1^n | n \geq 0\}$:
    $$S_1 \rightarrow 0S_11 \mid \varepsilon$$
  - Then create grammar for lang $\{1^n0^n | n \geq 0\}$:
    $$S_2 \rightarrow 1S_20 \mid \varepsilon$$
  - Then combine:
    $$S \rightarrow S_1 \mid S_2$$
    $$S_1 \rightarrow 0S_11 \mid \varepsilon$$
    $$S_2 \rightarrow 1S_20 \mid \varepsilon$$

New start variable & rule combines two smaller grammars

"|" = "or" = union (combines 2 rules with same left side)

Closed Operations on CFLs

- Start with small grammars and then combine (just like FSMs)
- "Or": $S \rightarrow S_1 \mid S_2$
- "Concatenate": $S \longrightarrow S_1 S_2$
- "Repetition": $S' \longrightarrow S' S_1 \mid \varepsilon$

In-class Example: Designing grammars

alphabet Σ is $\{0,1\}$

$\{w \mid w \text{ starts and ends with the same symbol}\}$

- $S \rightarrow \mathtt{0}C'\mathtt{0} \mid \mathtt{1}C'\mathtt{1} \mid \varepsilon$ — "string starts/ends with same symbol, middle can be anything"
- $C' \rightarrow C'C \mid \varepsilon$ — "middle: all possible terminals, repeated (ie, all possible strings)"
- $C \rightarrow \mathtt{0} \mid \mathtt{1}$ — "all possible terminals"

Next Time:

Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg expr <u>describes</u> a Regular lang	A CFG <u>describes</u> a CFL
Finite automaton (FSM)	???
An FSM <u>recognizes</u> a Regular lang	???

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A Reg expr <u>describes</u> a Regular lang	A CFG <u>describes</u> a CFL
Finite automaton (FSM)	Push-down automaton (PDA)
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Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg expr describes a Regular lang	A CFG describes a CFL
Finite automaton (FSM)	Push-down automaton (PDA)
An FSM recognizes a Regular lang	A PDA recognizes a CFL
DIFFERENCE:	**DIFFERENCE:**
A Regular lang is defined with a FSM	A CFL is defined with a CFG
Proved: Reg expr ⇔ Reg lang	*Must prove*: PDA ⇔ CFL

Check-in Quiz 2/16

On gradescope