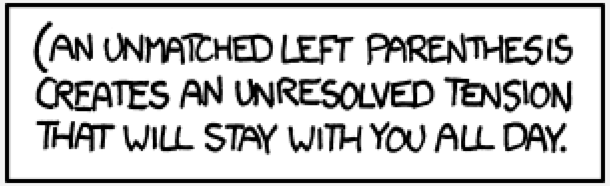
Deterministic CFLs, PDAs, and Parsing

Monday, February 28, 2022



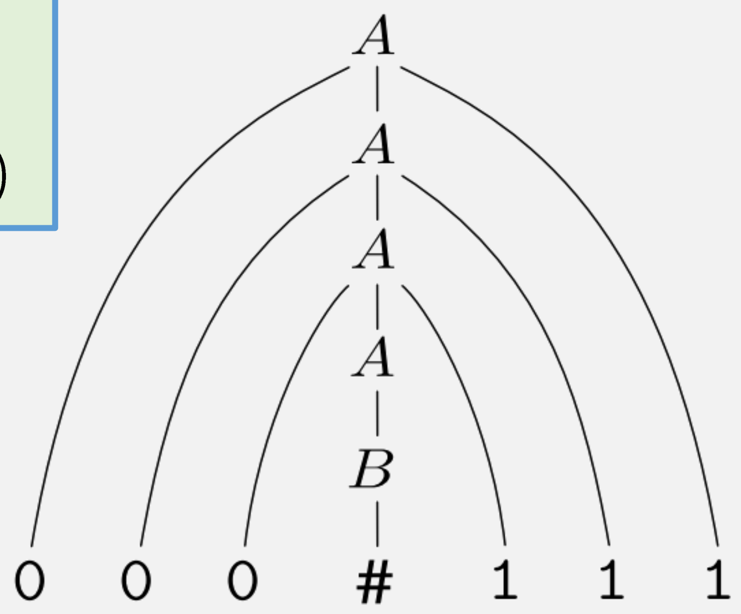
Announcements

- HW 4 in
- HW 5 out
  - Due Sun March 6 11:59pm
  - Problems about PDAs
- Upcoming: Spring Break is week of March 14

Previously: CFLs, CFGs, and Parse Trees

Generating strings:
- <u>Start</u> with *start variable*,
- <u>Repeatedly</u> *apply rules* to get a string (and **parse tree**)

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$



$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Today: Generating vs Parsing

Generating strings:
- <u>Start</u> with *start variable*,
- <u>Repeatedly</u> *apply rules* to get a string (and **parse tree**)

$$A \rightarrow 0A1$$
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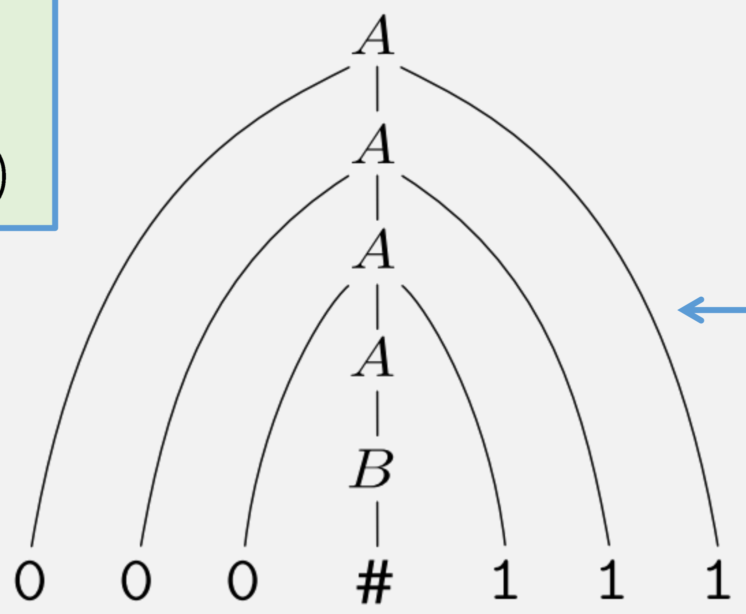


In practice, **the opposite is more interesting:** **<u>start with a string</u>,** **then parse it into parse tree**

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Generating vs Parsing

- In practice, **parsing** a string is more important than **generating** one
  - E.g., a **compiler's first step** parses source code into a parse tree
  - (Actually, *any* program with string inputs must first parse it)

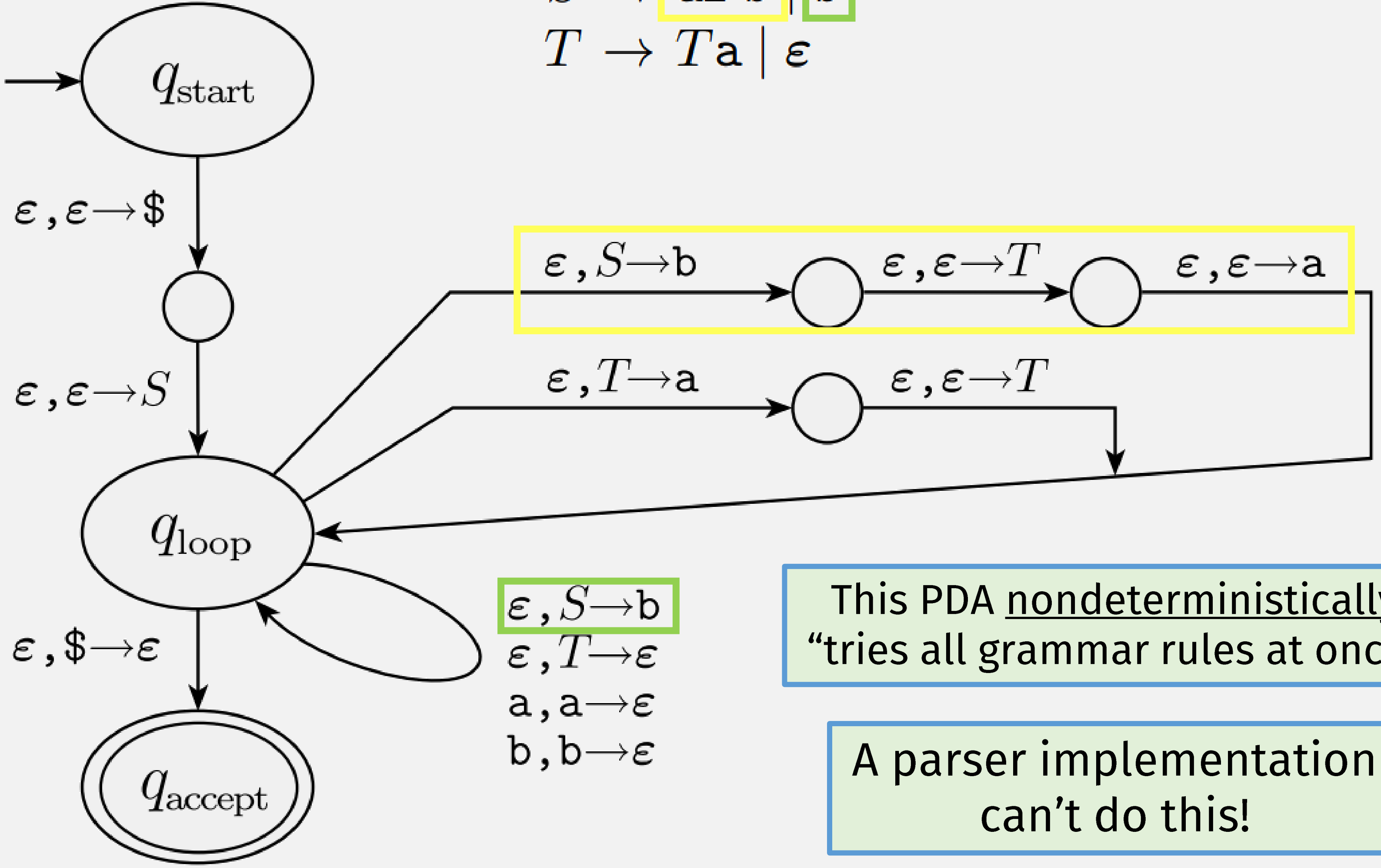
- But: the PDAs we've seen are <u>non-deterministic</u> (like NFAs)

- A compiler's parsing algorithm must be <u>deterministic</u>

- <u>So</u>: to model parsers, we need a **Deterministic PDA** (DPDA)

Last time: (Nondeterministic) PDA

$$S \to \texttt{a}T\texttt{b} \mid \texttt{b}$$
$$T \to T\texttt{a} \mid \varepsilon$$



This PDA <u>nondeterministically</u> "tries all grammar rules at once"

A parser implementation can't do this!

DPDA: Formal Definition

The language of a DPDA is called a ***deterministic context-free language***.

A ***deterministic pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta\colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow (Q \times \Gamma_\varepsilon) \cup \{\emptyset\}$ is the transition function
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

A ***pushdown automaton*** is a 6-tuple

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta\colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

Difference: DPDA has only **one possible action**, for any given state, input, and stack op (similar to **DFA** vs **NFA**)

This must take into account ε reads or stack ops! E.g., if $\delta(q, \mathrm{a}, X)$ is valid, then $\delta(q, \varepsilon, X)$ must not be

DPDAs are Not Equivalent to PDAs!

$$R \rightarrow S \mid T$$
$$S \rightarrow \mathtt{a}S\mathtt{b} \mid \mathtt{ab}$$
$$T \rightarrow \mathtt{a}T\mathtt{bb} \mid \mathtt{abb}$$

A PDA non-deterministically "tries all rules" (abandoning failed attempts) but a DPDA must decide on one rule at each step!

Should use S rule

aaabbb ⟶ aaSbb

Parsing = deriving reversed: start with string, end with parse tree

When parsing reaches this input position, which rule should it use, S or T?

Should use T rule

aaabbbbb ⟶ aaTbbbb

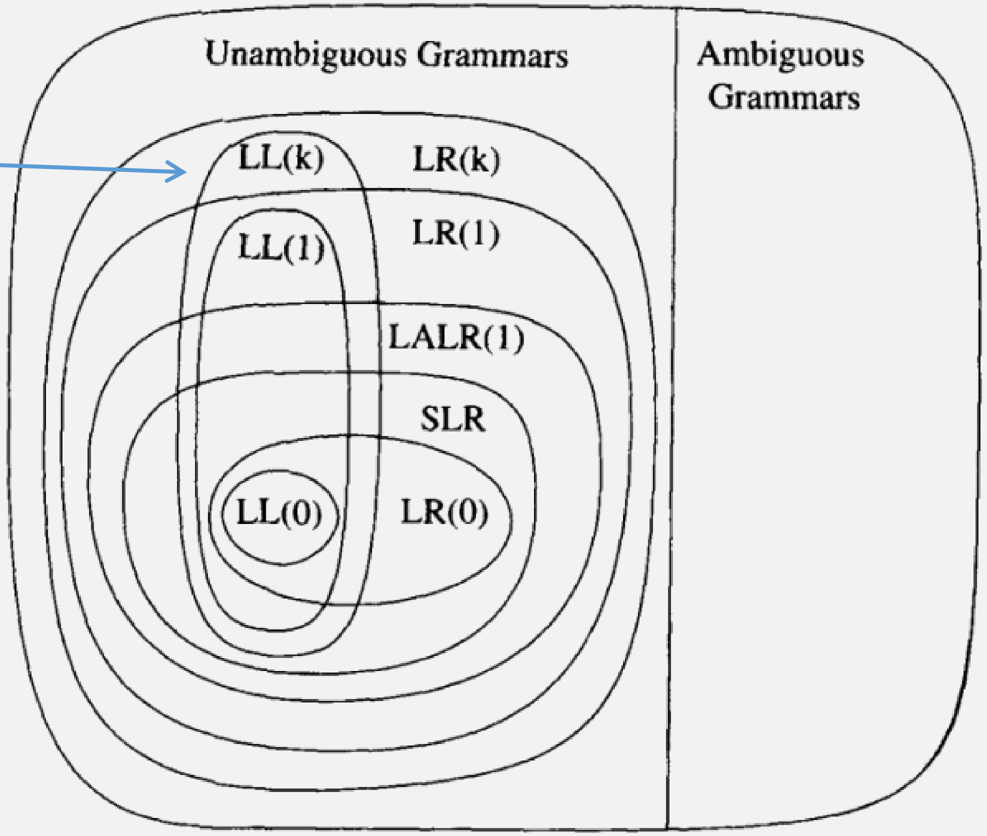
Don't know which rule to use because we can't see rest of the input!

PDAs recognize CFLs, but DPDAs only recognize DCFLs! (a subset of CFLs)

Subclasses of CFLs

DCFLs

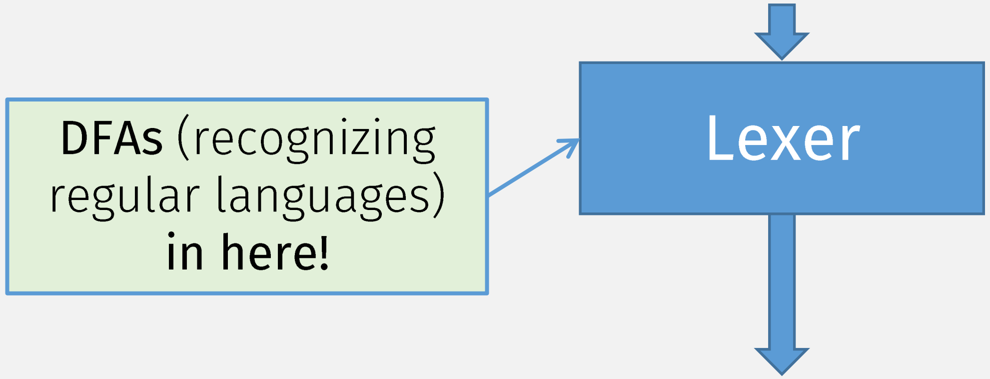
Programming language parsers / compilers are ideally in here



All CFLS

Compiler Stages

A program string (chars) (e.g., `a : = ( 5 + 3 ) ;` ...)



Program "words"
(e.g., `ID(a) ASSIGN LPAREN NUM(5) PLUS NUM(3) RPAREN SEMI` ...)

A Lexer Implementation

```
%{
/* C Declarations: */
#include "tokens.h"    /* definitions of IF, ID, NUM, ... */
#include "errormsg.h"
union {int ival; string sval; double fval;} yylval;
int charPos=1;
#define ADJ  (EM_tokPos=charPos, charPos+=yyleng)
%}
/* Lex Definitions: */
digits  [0-9]+
%%
/* Regular Expressions and Actions: */
if                       {ADJ; return IF;}
[a-z][a-z0-9]*           {ADJ; yylval.sval=String(yytext);
                           return ID;}
{digits}                 {ADJ; yylval.ival=atoi(yytext);
                           return NUM;}
({digits}"."[0-9]*)|([0-9]*"."{digits})     {ADJ;
                           yylval.fval=atof(yytext);
                           return REAL;}
("--"[a-z]*"\n")|(" "|"\n"|"\t")+     {ADJ;}
.                        {ADJ; EM_error("illegal character");}
```

DFAs (represented as **regular expressions**)!

A “`lex`” tool translates this to a (C program) implementation of a lexer

Compiler Stages

A program (chars) (e.g., `a := ( 5 + 3 ) ;` ...)

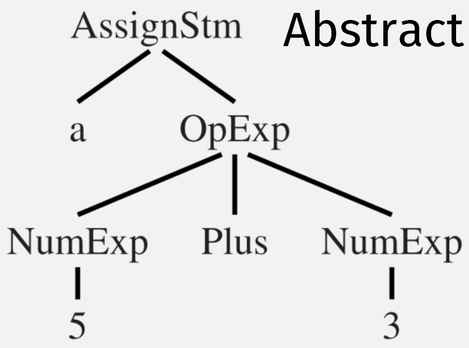
DFAs (recognizing regular languages) in here!

Lexer

Program “words”
(e.g., `ID(a) ASSIGN LPAREN NUM(5) PLUS NUM(3) RPAREN SEMI` ...)

DPDAs (recognizing DCFLs) in here!

Parser



Abstract Syntax tree (AST), i.e., a **parse tree!**

A Parser Implementation

```
%{
int yylex(void);
void yyerror(char *s) { EM_error(EM_tokPos, "%s", s); }
%}
%token ID WHILE BEGIN END DO IF THEN ELSE SEMI ASSIGN
%start prog
%%

prog: stmlist

stm : ID ASSIGN ID
    | WHILE ID DO stm
    | BEGIN stmlist END
    | IF ID THEN stm
    | IF ID THEN stm ELSE stm

stmlist : stm
        | stmlist SEMI stm
```

Just write the CFG!

A "yacc" tool translates this to a (C program) implementation of a parser

Parsing

$$
\begin{aligned}
R &\to S \mid T \\
S &\to \mathtt{a}S\mathtt{b} \mid \mathtt{ab} \\
T &\to \mathtt{a}T\mathtt{bb} \mid \mathtt{abb}
\end{aligned}
$$

$$\mathtt{aa}\underline{\mathtt{ab}}\mathtt{bb} \rightarrowtail \mathtt{a}\underline{\mathtt{aSb}}\mathtt{b}$$

A parser must be able to <u>choose the one correct rule</u>, when reading input left-to-right

$$\mathtt{aa}\underline{\mathtt{abb}}\mathtt{bbb} \rightarrowtail \mathtt{a}\underline{\mathtt{aTbb}}\mathtt{bb}$$

LL parsing

- **L** = left-to-right
- **L** = leftmost derivation

Game: "You're the Parser":
Guess which rule applies?

1 $S \to \text{if } E \text{ then } S \text{ else } S$

2 $S \to \text{begin } S\ L$

3 $S \to \text{print } E$

4 $L \to \text{end}$

5 $L \to ;\ S\ L$

6 $E \to \text{num} = \text{num}$

```
if 2 = 3 begin print 1; print 2; end else print 0
```

LL parsing

- **L** = left-to-right
- **L** = leftmost derivation

1 $S \rightarrow \text{if } E \text{ then } S \text{ else } S$

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if 2 = 3 begin print 1; print 2; end else print 0
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LL parsing

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6 $E \rightarrow \text{num} = \text{num}$

```
if 2 = 3 begin print 1; print 2; end else print 0
```

"Prefix" languages (like Scheme/Lisp) are easily parsed with LL parsers

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

1 $S \rightarrow S\ ;\ S$
2 $S \rightarrow \text{id} := E$
3 $S \rightarrow \text{print}\ (\ L\ )$
4 $E \rightarrow \text{id}$
5 $E \rightarrow \text{num}$
6 $E \rightarrow E + E$

```
a := 7;
b := c + (d := 5 + 6, d)
```

When parse is here, can't determine whether it's an assign (:=) or addition (+)

Need to save input to some temporary memory, like a **stack**: this is a job for a (D)PDA!!

Stack	*Input*	*Action*
1	a := 7 ; b := c + (d := 5 + 6 , d) $	*shift*
1 id_4	:= 7 ; b := c + (d := 5 + 6 , d) $	*shift*
1 id_4 $:=_6$	7 ; b := c + (d := 5 + 6 , d) $	*shift*
1 id_4 $:=_6$ num_{10}	; b := c + (d := 5 + 6 , d) $	*reduce* $E \rightarrow \text{num}$
1 id_4 $:=_6$ E_{11}	; b := c + (d := 5 + 6 , d) $	*reduce* $S \rightarrow \text{id} := E$
1 S_2	; b := c + (d := 5 + 6 , d) $	*shift*

push

State name

"push"

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

$$
\begin{array}{ll}
S \rightarrow S\ ;\ S & E \rightarrow \text{id} \\
S \rightarrow \text{id}\ :=\ E & E \rightarrow \text{num} \\
S \rightarrow \text{print}\ (\ L\ ) & E \rightarrow E\ +\ E
\end{array}
$$

Stack	*Input*	*Action*
$_1$	a := 7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$	:= 7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$:=$_6$	7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$:=$_6$ num$_{10}$	; b := c + (d := 5 + 6 , d) $	*reduce* $E \rightarrow$ num
$_1$ id$_4$:=$_6$ E_{11}	; b := c + (d := 5 + 6 , d) $	*reduce* $S \rightarrow$ id:=E
$_1$ S_2	; b := c + (d := 5 + 6 , d) $	*shift*

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

$$S \rightarrow S\ ;\ S \qquad E \rightarrow \text{id}$$
$$S \rightarrow \text{id} := E \qquad E \rightarrow \text{num}$$
$$S \rightarrow \text{print}\ (\ L\ ) \qquad E \rightarrow E + E$$

Stack	*Input*	*Action*
$_1$	`a := 7 ; b := c + ( d := 5 + 6 , d ) $`	*shift*
$_1\ \text{id}_4$	`:= 7 ; b := c + ( d := 5 + 6 , d ) $`	*shift*
$_1\ \text{id}_4\ :=_6$	`7 ; b := c + ( d := 5 + 6 , d ) $`	*shift*
$_1\ \text{id}_4\ :=_6\ \text{num}_{10}$	`; b := c + ( d := 5 + 6 , d ) $`	*reduce* $E \rightarrow$ num
$_1\ \text{id}_4\ :=_6\ E_{11}$	`; b := c + ( d := 5 + 6 , d ) $`	*reduce* $S \rightarrow$ id$:=E$
$_1\ S_2$	`; b := c + ( d := 5 + 6 , d ) $`	*shift*

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

1. $S \rightarrow S \;;\; S$
2. $S \rightarrow \text{id} := E$
3. $S \rightarrow \text{print} (L)$
4. $E \rightarrow \text{id}$
5. $E \rightarrow \text{num}$
6. $E \rightarrow E + E$

Stack	*Input*	*Action*
$_1$	a := 7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$	:= c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$:=$_6$	:= c + (d := 5 + 6 , d) $	*shift*
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Can determine (rightmost) rule

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

1 $S \rightarrow S \; ; \; S$

2 $S \rightarrow \text{id} := E$

3 $S \rightarrow \text{print} \; (\; L \;)$

4 $E \rightarrow \text{id}$

5 $E \rightarrow \text{num}$

6 $E \rightarrow E + E$

Stack	*Input*	*Action*
$_1$	a := 7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$	:= 7 ; b := c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$:=$_6$	= c + (d := 5 + 6 , d) $	*shift*
$_1$ id$_4$:=$_6$ num$_{10}$	= c + (d := 5 + 6 , d) $	*reduce* $E \rightarrow$ num
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Can determine (rightmost) rule

LR parsing

- **L** = left-to-right
- **R** = rightmost derivation

$$S \rightarrow S \,;\, S \qquad E \rightarrow \text{id}$$
$$S \rightarrow \text{id} := E \qquad E \rightarrow \text{num}$$
$$S \rightarrow \text{print} \,(\, L \,) \qquad E \rightarrow E + E$$

Stack	*Input*	*Action*
$_1$	a := 7 ; b := c + (d := 5 + 6 , d) $	*shift*
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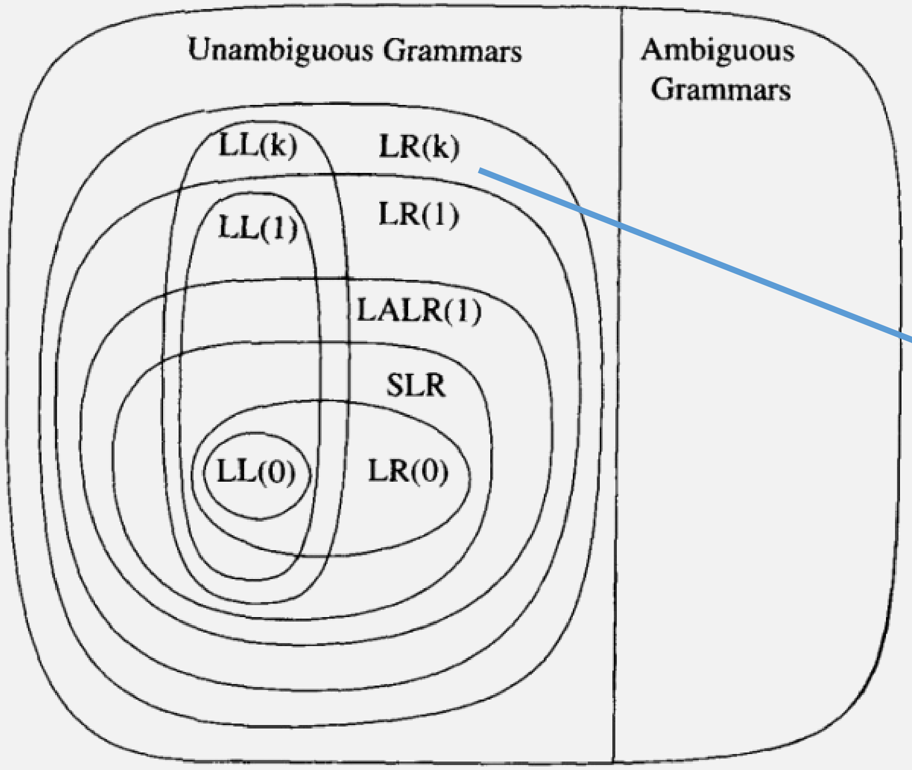
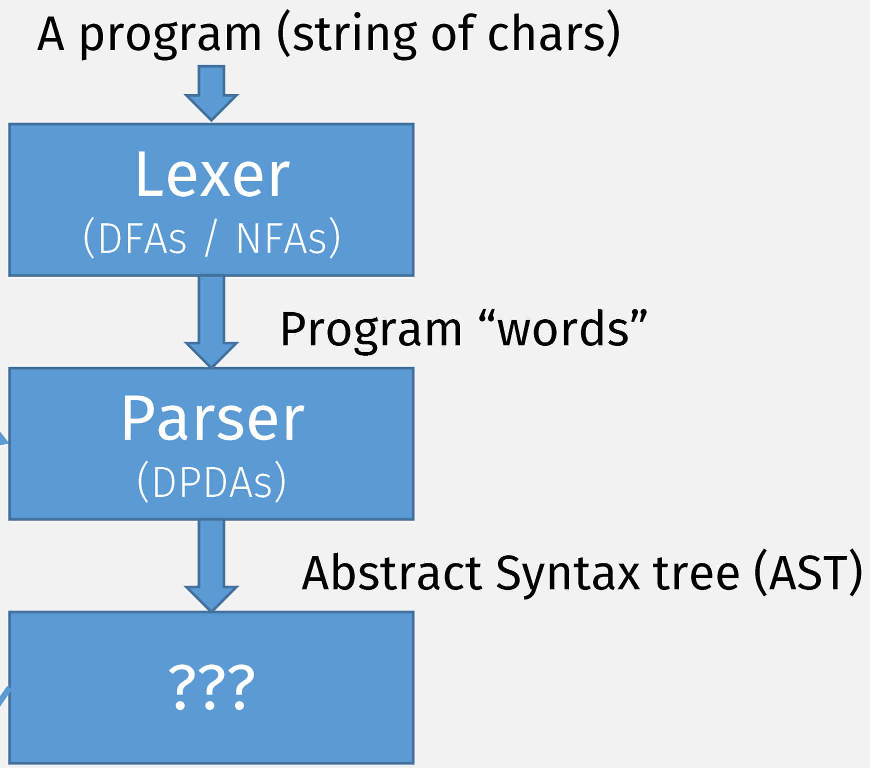
To learn more, take a Compilers Class!





This phase needs computation that goes beyond CFLs

Non-CFLs

Flashback: Pumping Lemma for Regular Langs

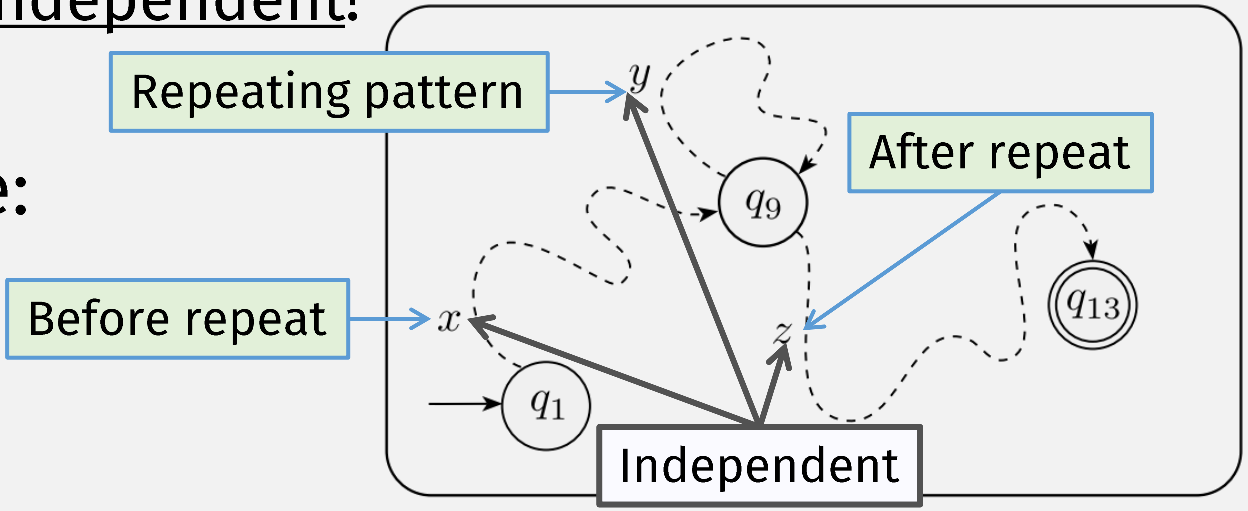
- The Pumping Lemma <u>describes how strings **repeat**</u>

- Regular language strings can (only) repeat using Kleene pattern
  - But the <u>substrings are independent</u>!

- A non-regular language:

$$\{0^n 1^n \mid n \geq 0\}$$

Kleene star can't express this pattern:
2nd part <u>depends</u> on (length of) 1st part



- Q: How do CFLs repeat?

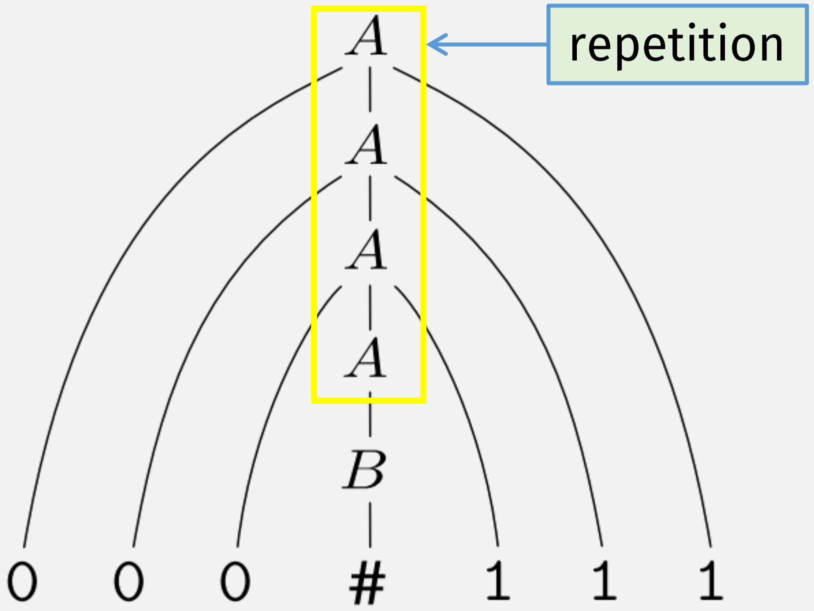
Repetition and Dependency in CFLs

Parts before/after repetition point are linked

Repetition

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

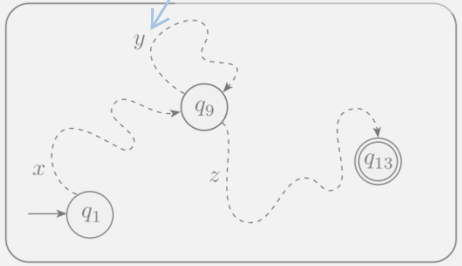
$$\{0^n\#1^n \mid n \geq 0\}$$



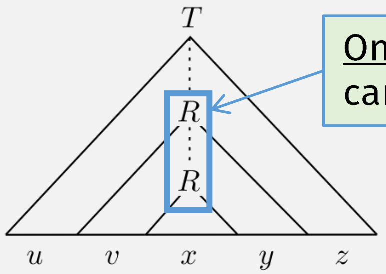
$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

How Do Strings in CFLs Repeat?

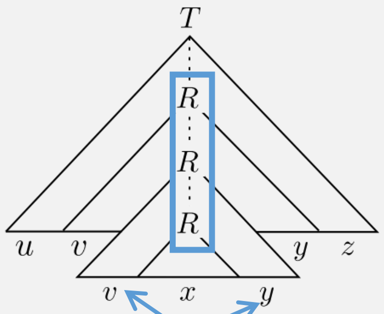
NFA can take loop transition any number of times, to process repeated y in input



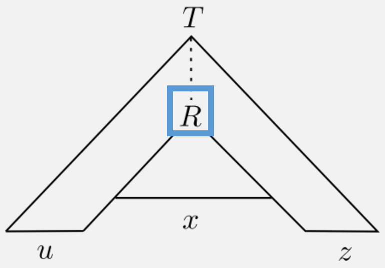
- Strings in regular languages repeat states
- Strings in CFLs repeat subtrees in the parse tree



One repeated subtree means that it can be repeated any number of times



Linked parts

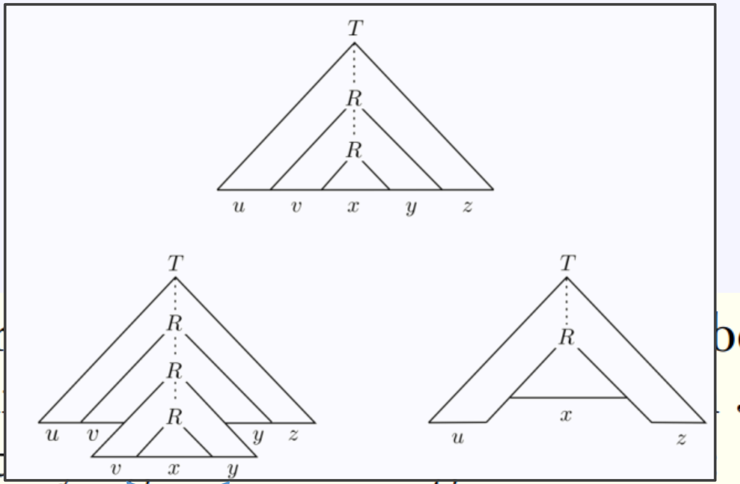


Pumping Lemma for CFLS

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Now there are <u>two</u> pumpable parts.
But they must be <u>pumped together</u>!



Pumping lemma If A is a regular ... ber p (the pumping length) where if s is any stri... s may be divided into three pieces, $s = xyz$, sat...

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Two pumpable parts, pumped together

Non CFL example: $D = \{ww|\ w \in \{0,1\}^*\}$

Previous: D is nonregular: unpumpable counterexample s: 0^p10^p1

Now: this s **can** be pumped according to CFL pumping lemma:

$$\overbrace{\underbrace{000\cdots000}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{x}}^{0^p1}\ \overbrace{\underbrace{0}_{y}\ \underbrace{000\cdots0001}_{z}}^{0^p1}$$

Pumping v and y (together) produces string still in D

- CFL Pumping Lemma conditions:
  ☑ **1.** for each $i \geq 0$, $uv^ixy^iz \in A$,
  ☑ **2.** $|vy| > 0$, and
  ☑ **3.** $|vxy| \leq p$.

This doesn't prove that the language is a CFL!
It only means that this attempt to prove that the language is not a CFL failed.

Non CFL example: $D = \{ww \mid w \in \{0,1\}^*\}$

- Need another counterexample string *s*:

If *vyx* is contained in first or second half, then any pumping will break the match ✖

$$0^p 1^p 0^p 1^p$$

So *vyx* must straddle the middle ✖

But any pumping still breaks the match because order is wrong

- CFL Pumping Lemma conditions:

1. for each $i \geq 0$, $uv^ixy^iz \in A$,

2. $|vy| > 0$, and

3. $|vxy| \leq p$.

Now we have proven that **this language is not a CFL!**

CFL Pumping Lemma is Too Restrictive?

???

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

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Review: Regular Language Pumping Lemma

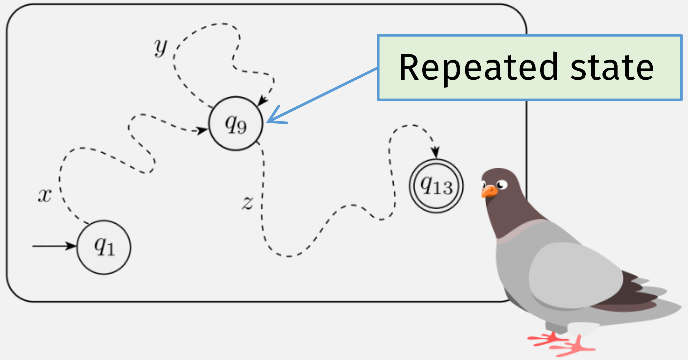
- The pumping length p for a language L is …

  … the # of states in that language's NFA!

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

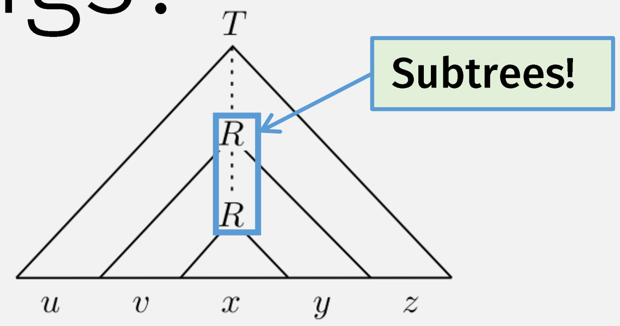
- If string length > # of states, then some state must repeat



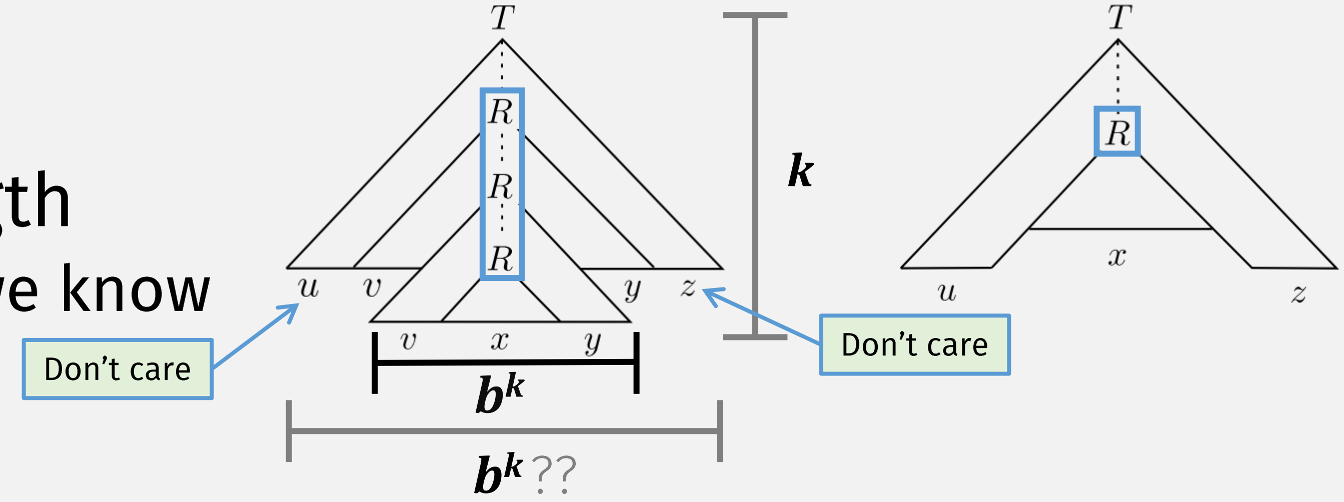
- If a state is repeated once, then it can repeat multiple times

Repeating Pattern in CFL Strings?

- When are we guaranteed to have a repeated subtree?
  - When height of parse tree > # of rules!



- Let k = # of rules and b = longest rule RHS length
  - Then the length string where we know there's a repeat is b^k
  - I.e., pumping length = b^k???



Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Pumping Length could be too short!

A Pumpable Non-CFL?

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

- CFL Pumping Lemma says:
  - *"All CFLs are pumpable"*
  - So if we find a non-pumpable language … it's not a CFL!
- Pumping Lemma does not say:
  - *"All nonCFLs are not pumpable"*
  - (statement != it's inverse)
  - So Pumping Lemma might not be able to prove some non-CFLs!

Example:

$$L = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k\mathtt{d}^l \mid i = 0 \text{ or } j = k = l\}$$

- For any counterexample, split into *uvxyz* where,
  - v = first char
  - z = remaining chars
  - $u = x = y = \varepsilon$
- If there are **a**s …
  - … it's pumpable bc # of as is arbitrary
- If there there are no **a**s
  - … it's pumpable bc # of other chars is arbitrary

This language is pumpable … but not a CFL!
(can't come up with a CFG)

Ogden's Lemma (generalizes pumping lemma)

Ogden's lemma is: If L is a CFL, then there is a constant n, such that if z is any string of length at least n in L, in which we select at least n positions to be *distinguished*, then we can write $z = uvwxy$, such that:

1. vwx has at most n distinguished positions.
2. vx has at least one distinguished position.
3. For all i, uv^iwx^iy is in L.

Says that every long enough <u>segment</u> must be pumpable

<u>Example</u>:

$$L = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k\mathtt{d}^l \mid i = 0 \text{ or } j = k = l\}$$

This language is not a CFL because it doesn't satisfy Ogden's Lemma

<u>Counterexample</u>: $\mathtt{ab}^n\mathtt{c}^n\mathtt{d}^n$

- n "distinguished" positions must include non-`a` character
  - Impossible to pump no matter which n chars are chosen

A Practical Non-CFL

- **XML**
  - ELEMENT → <TAG>CONTENT</TAG>
  - Where TAG is any string

- XML also looks like this non-CFL: $D = \{ww \mid w \in \{0,1\}^*\}$

- This means XML is not context-free!
  - Note: HTML *is* context-free because …
  - … there are only a finite number of tags,
  - so they can be embedded into a finite number of rules.

- In practice:
  - XML is parsed as a CFL, with a CFG
  - Then matching tags checked in a 2nd pass with a more powerful machine …

Next Time: A More Powerful Machine …

M_1 accepts its input if it is in language: $B = \{w\#w \mid w \in \{0,1\}^*\}$

M_1 = "On input string w:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.

Infinite memory, initially starts with input

Can move to, and read/write from, arbitrary memory locations

In-class quiz 2/28

See gradescope