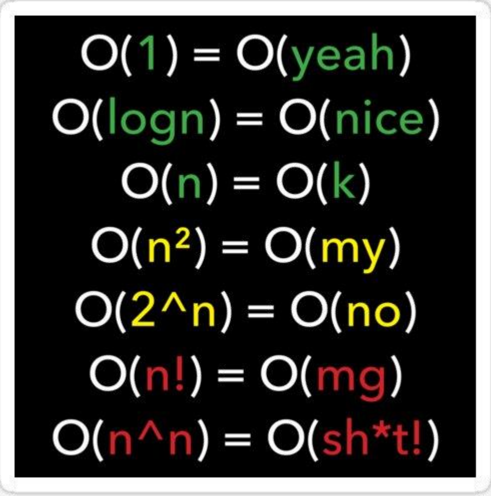
UMB CS420

Polynomial Time (P)

Wednesday, April 20, 2022



Announcements

- HW 9 in
  - Mapping Reducibility and Unrecognizability
  - ~~Due Tues 4/19 11:59pm EST~~

- HW 10 out
  - Time Complexity and Poly time
  - Due Tues 4/26 11:59pm EST

Last Time: Time Complexity

Running Time / **Time Complexity** is a property of decider TMs (algorithms)

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Depends on size of input

Worst case

Last Time: Time Complexity Classes

Big-O = asymptotic upper bound,
i.e., "only care about **large** n"

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(t(n))$, to be the collection of all $\boxed{\text{languages}}$ that are decidable by an $\boxed{O(t(n))}$ time Turing machine.

Remember:
- <u>TMs</u> have a **time complexity** (i.e., a running time),
- <u>languages</u> are in a **time complexity class**

The **time complexity class** of a <u>language</u> is determined by the **time complexity** (running time) of its deciding <u>TM</u>

A <u>language</u> can have <u>multiple</u> deciding TMs, so could be in <u>multiple</u> **time complexity classes**

The Polynomial Time Complexity Class (**P**)

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

- Corresponds to "realistically" solvable problems:
  - Problems in **P**
    - = "solvable" or "tractable"
  - Problems outside **P**
    - = "unsolvable" or "intractable"

"Unsolvable" Problems

Mathematicians are weird.

- Unsolvable problems (those outside **P**):
  - usually only have "brute force" solutions
  - i.e., "try all possible inputs"
  - "unsolvable" applies only to large ***n***

Brute-force attack

From Wikipedia, the free encyclopedia

In cryptography, a **brute-force attack** consists of an attacker submitting many passwords or passphrases with the hope of eventually guessing a combination correctly. The attacker systematically checks all possible passwords and passphrases until the correct one is found. Alternatively, the attacker can attempt to guess the key which is typically created from the password using a key derivation function. This is known as an **exhaustive key search**.

Amount of Time to Crack Passwords

"abcdefg" 7 characters	.29 milliseconds
"abcdefgh" 8 characters	5 hours
"abcdefghi" 9 characters	5 days
"abcdefghij" 10 characters	4 months
"abcdefghijk" 11 characters	1 decade
"abcdefghijkl" 12 characters	2 centuries

today → How to prove something is "solvable"?

How to prove something is "unsolvable"?

3 Problems in **P**

- A <u>Graph</u> Problem:

  $PATH = \{\langle G, s, t\rangle |\ G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A <u>Number</u> Problem:

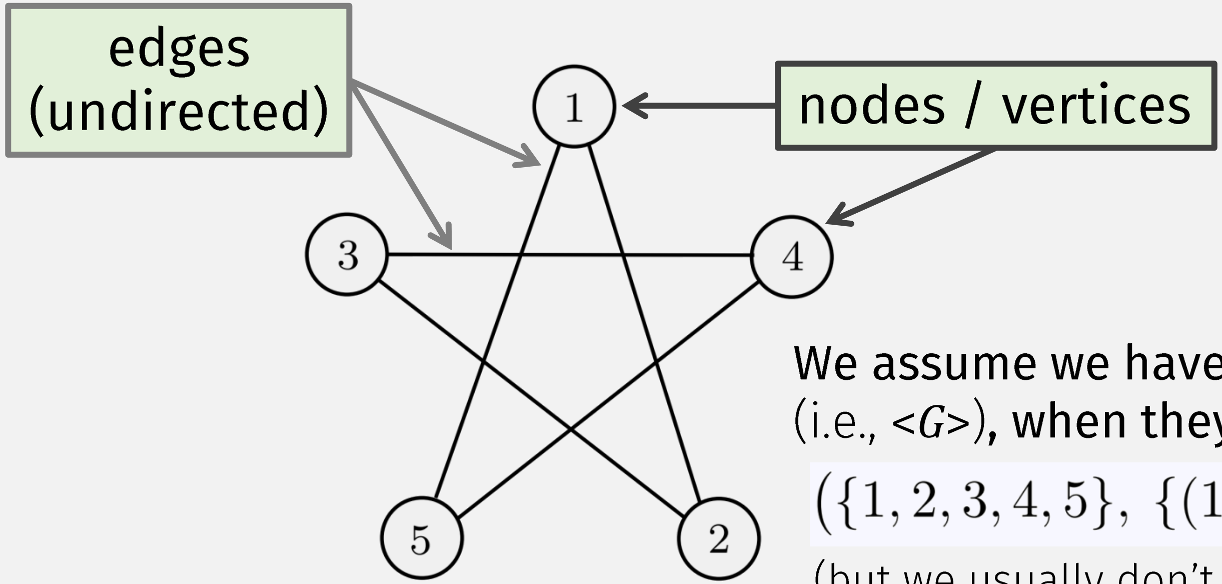
  $RELPRIME = \{\langle x, y\rangle |\ x \text{ and } y \text{ are relatively prime}\}$

- A <u>CFL</u> Problem:

  Every context-free language is a member of P

- To prove that a language is "solvable", i.e., in **P** …
  - … construct a polynomial time algorithm deciding the language
- (These may also have nonpolynomial, i.e., brute force, algorithms)
  - Check all possible … paths/numbers/strings …

Interlude: Graphs (see Sipser Chapter 0)



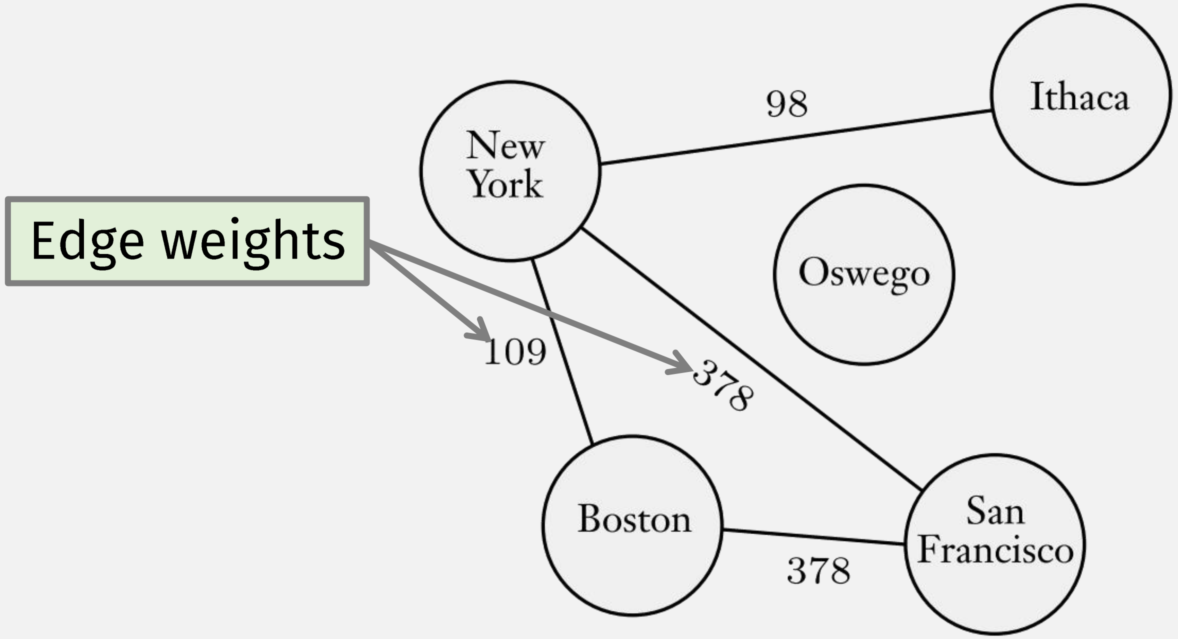
We assume we have *some* **string encoding of a graph** (i.e., <*G*>)**, when they are args to TMs,** e.g.:

$$\big(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\}\big)$$

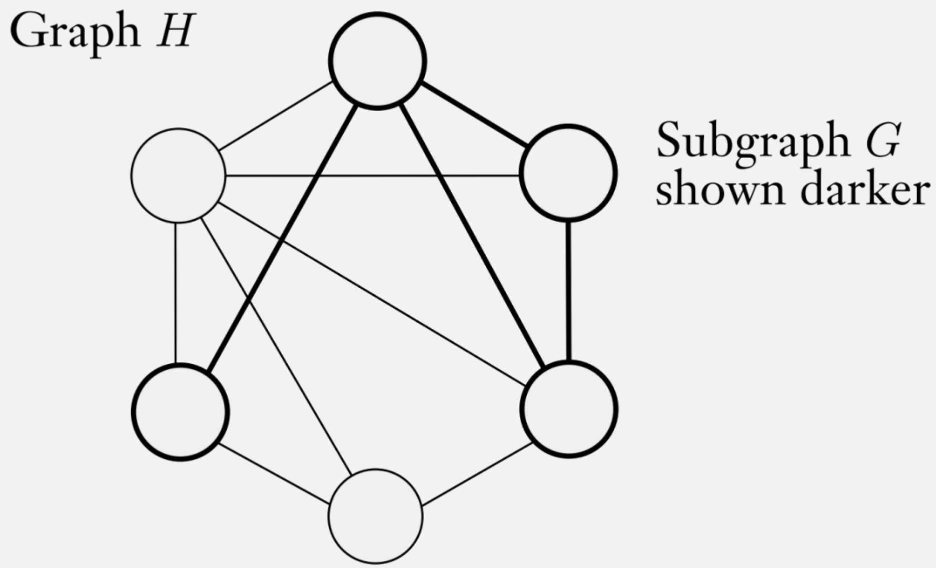
(but we usually don't care about the actual details)

- **Edge defined by two nodes** (order doesn't matter)
- Formally**, a graph = a pair (***V*, *E***)**
  - Where ***V*** **= a set of nodes,** ***E*** **= a set of edges**

Interlude: Weighted Graphs

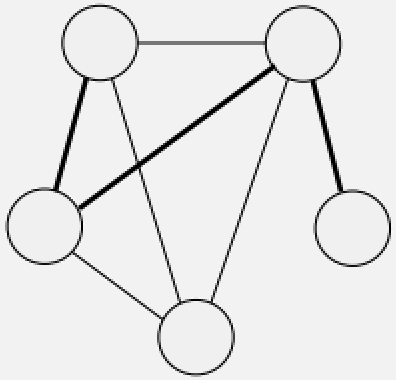


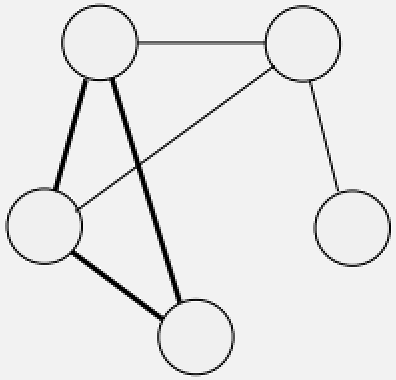
Interlude: Subgraphs

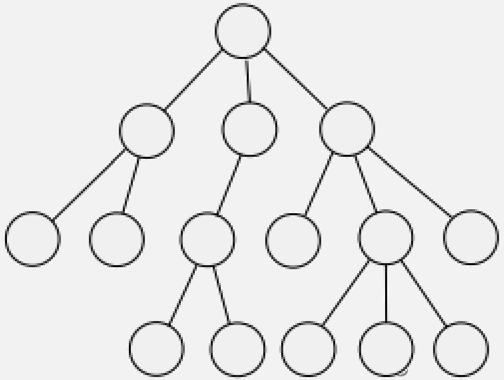


<u>Interlude</u>: Paths and other Graph Things

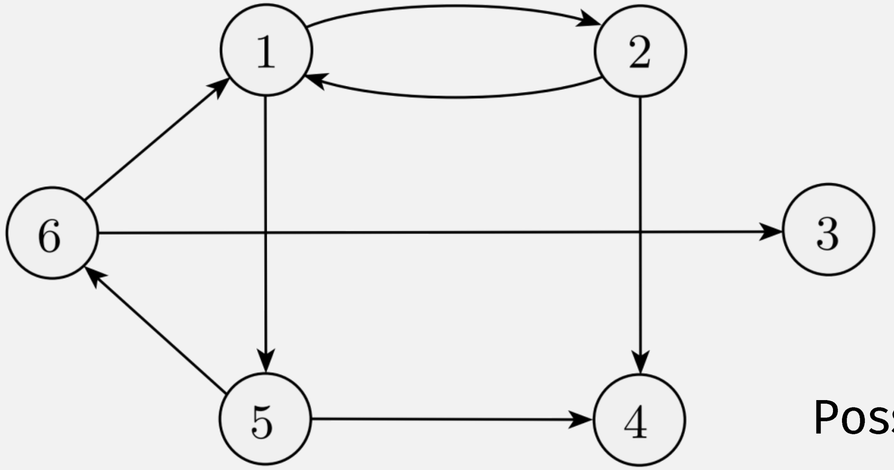
- **Path**
  - A sequence of nodes connected by edges
- **Cycle**
  - A path that starts/ends at the same node
- **Connected graph**
  - Every two nodes has a path
- **Tree**
  - A connected graph with no cycles







Interlude: Directed Graphs



Possible **string encoding** given to TMs:

$(\{1,2,3,4,5,6\},\ \{(1,2), (1,5), (2,1), (2,4), (5,4), (5,6), (6,1), (6,3)\})$

Each pair of nodes included twice

- **Directed graph** = (V, E)
  - V = set of nodes, E = set of edges
- An **edge** is a pair of nodes (u,v), *order now matters*
  - u = "from" node, v = "to" node
- "degree" of a node: number of edges connected to the node
  - Nodes in a directed graph have both indegree and outdegree

Interlude: Graph Encodings

$$(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\})$$

- For graph algorithms, "length of input" n usually = # of vertices
  - (Not number of chars in the encoding)
- So given graph $G = (V, E)$, $n = |V|$
- Max edges?
  - $= O(|V|^2) = O(n^2)$
- So if a set of graphs (call it lang L) is decided by a TM where
  - # steps of the TM = polynomial in the # of vertices

    Or polynomial in the # of edges
  - Then L is in $\mathbf{P}$

3 Problems in **P**

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A Number Problem:

  $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- A CFL Problem:

  Every context-free language is a member of P

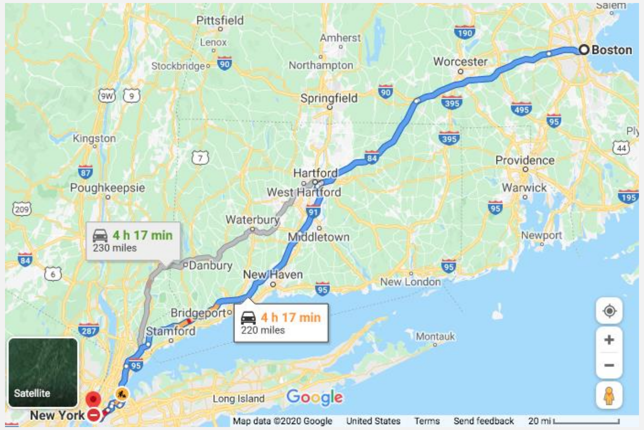
P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

A Graph Theorem: $PATH \in \mathrm{P}$

$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

(A **path** is a sequence of nodes connected by edges)



- To prove that a language is in **P** ...

- ... we must construct a polynomial time algorithm deciding the lang

- A non-polynomial (i.e., "brute force") algorithm:
  - check all possible paths,
  - see if any connect s to t
  - If n = # vertices, then # paths $\approx n^n$

A Graph Theorem: $PATH \in \mathrm{P}$

$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

PROOF A polynomial time algorithm M for $PATH$ operates as follows.

M = "On input $\langle G, s, t\rangle$, where G is a directed graph with nodes s and t:

1. Place a mark on node s.
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of G. If an edge (a, b) is found going from a marked node a to an unmarked node b, mark node b.
4. If t is marked, *accept*. Otherwise, *reject*."

# of steps (worst case) **(n = # nodes):**

- Line 1: **1 step**

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# of steps (worst case) **(n = # nodes):**

- Line 1: **1 step**
- Lines 2-3 (loop):
  - Steps/iteration (line 3): max # steps = max # edges = $O(n^2)$

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  - ➢ Total: $\boxed{O(n^3)}$

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$\boxed{O(\boldsymbol{n}^3)}$

(Breadth-first search)

# of steps (worst case) **($\boldsymbol{n}$ = # nodes):**

- Line 1: **1 step**
- Lines 2-3 (loop):
  - Steps/iteration (line 3): max # steps = max # edges = $O(\boldsymbol{n}^2)$
  - # iterations (line 2): loop runs at most $\boldsymbol{n}$ times
  - Total: $O(\boldsymbol{n}^3)$
- Line 4: **1 step**

➢ Total = $1 + 1 + O(\boldsymbol{n}^3) = O(\boldsymbol{n}^3)$

3 Problems in **P**

☑ • A Graph Problem:

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• A Number Problem:

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• A CFL Problem:

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A Number Theorem: $RELPRIME \in \mathrm{P}$

$RELPRIME = \{\langle x, y \rangle |\ x \text{ and } y \text{ are relatively prime}\}$

- Two numbers are **relatively prime:** if their $\gcd = 1$
  - $\gcd(x, y)$ = largest number that divides both x and y
  - E.g., $\gcd(8, 12) = 4$
- Brute force (exponential) algorithm deciding $RELPRIME$:
  - Try all of numbers (up to x or y), see if it can divide both numbers
  - Q: Why is this exponential?
  - HINT: What is a typical "representation" of numbers?
  - A: binary numbers
  - (if $x = 2^n$, then trying x numbers is exponential in n, the number of digits)
- A $\gcd$ algorithm that runs in polynomial time:
  - Euclid's algorithm

A GCD Algorithm for: *RELPRIME* $\in$ P

$RELPRIME = \{\langle x, y \rangle | \ x \text{ and } y \text{ are relatively prime}\}$

Modulo
(i.e., **remainder**)
cuts *x* (at least) **in half**

15 mod 8 = 7
17 mod 8 = 1

Cutting *x* in half
every step requires:
log *x* steps

The Euclidean algorithm E is as follows.

E = "On input $\langle x, y \rangle$, where x and y are natural numbers in binary:
1. Repeat until $y = 0$:
2. Assign $x \leftarrow x \bmod y$.
3. Exchange x and y.
4. Output x."

***O*(*n*)**

Each number is
cut in half every
other iteration

Total run time (assume $x > y$): $2\log x = 2\log 2^n = \boldsymbol{O(n)}$,
where n = number of binary digits in (ie length of) x

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Review: A Decider for Any CFL

Given any CFL L, with CFG G, the following decider M_G decides L:

M_G = "On input w:
1. Run TM S on input $\langle G, w\rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

M_G is a decider, bc S is a decider

S is a decider for: $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$

M_G accepts all $w \in L$, for any CFL L

S = "On input $\langle G, w\rangle$, where G is a CFG and w is a string:
1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

Therefore, every CFL is decidable

But, is every CFL decidable in poly time?

A Decider for Any CFL: Running Time

Given any CFL L, with CFG G, the following decider M_G decides L:

M_G = "On input w:
1. Run TM S on input $\langle G, w \rangle$.
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This algorithm runs in exponential time

Worst case:
$|R|^{2n-1}$ steps = $O(2^n)$
(R = set of rules)

A CFL Theorem: Every context-free language is a member of P

- Given a CFL, we must construct a decider for it …
- … that runs in polynomial time

Dynamic Programming

- Keep track of partial solutions, and re-use them
  - Start with smallest and build up
- For CFG problem, instead of re-generating entire string …
  - … keep track of substrings generated by each variable

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This duplicates a lot of work because many strings might have the same beginning derivation steps

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b					
a					
a					
b					
a					

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	b	**a**	**a**	**b**	**a**
b	vars generating "b"	vars for "ba"	vars for "baa"	…	
a		vars for "a"	vars for "aa"	vars for "aab"	
a			…		
b					
a					

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Algo:
- **For** each single char c and var A:
  - **If** A → c is a rule, add A to table

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Substring start char

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a		vars for "a"	vars for "aa"	vars for "aab"	
a			…		
b					
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b	B				
a		A,C			
a			A,C		
b				B	
a					A,C

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<u>Algo</u>:
- For each single char c and var A:
  - If A $\rightarrow$ c is a rule, add A to table
- For each substring s (len > 1):
  - For each split of substring s into x,y:
    - For each rule of shape A $\rightarrow$ BC:
      - Use table to check if B generates x and C generates y

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b	B				
a		A,C			
a			A,C		
b				B	
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  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B

For substring "**ba**", split into "**b**" and "**a**":
- For rule S → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO
- For rule S → BC
  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
  - **YES**
- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

Substring end char

Substring start char

	b	**a**	**a**
b	B		
a		A,C	
a			A,C
b			
a			

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Substring start char

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b	B	S,A	
a		A,C	
a			A,C
b			
a			

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- For rule S → AB
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  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
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- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

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Algo: For each char, var …

- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring s For each substring, split, rule …
  - For each split of substring s into x,y:
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Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b	B	S,A			S,A,C
a		A,C	B	B	S,A,C
a			A,C	S,C	B
b				B	S,A
a					A,C

If S is here, accept

A CFG Theorem: Every context-free language is a member of P

D = "On input $w = w_1 \cdots w_n$:

1. For $w = \varepsilon$, if $S \to \varepsilon$ is a rule, *accept*; else, *reject*. 〚 $w = \varepsilon$ case 〛
2. For $i = 1$ to n: $O(\boldsymbol{n})$ 〚 examine each substring of length 1 〛
3. For each variable A: *#vars*
4. Test whether $A \to \mathtt{b}$ is a rule, where $\mathtt{b} = w_i$.
5. If so, place A in $table(i, i)$.
6. For $l = 2$ to n: $O(\boldsymbol{n})$ 〚 l is the length of the substring 〛
7. For $i = 1$ to $n - l + 1$: $O(\boldsymbol{n})$ 〚 i is the start position of the substring 〛
8. Let $j = i + l - 1$. 〚 j is the end position of the substring 〛
9. For $k = i$ to $j - 1$: $O(\boldsymbol{n})$ 〚 k is the split position 〛
10. For each rule $A \to BC$: *#rules*
11. If $table(i, k)$ contains B and $table(k + 1, j)$ contains C, put A in $table(i, j)$.
12. If S is in $table(1, n)$, *accept*; else, *reject*."

For each:
- char
- var

#vars * n = $O(\boldsymbol{n})$

For each:
- substring
- split
- rule

#rules * $O(\boldsymbol{n})$ * $O(\boldsymbol{n})$ * $O(\boldsymbol{n})$ = $O(\boldsymbol{n}^3)$

Total: $\boxed{O(\boldsymbol{n}^3)}$

(This is also known as the Earley parsing algorithm)

Summary: 3 Problems in $\mathbf{P}$

- ☑ • A Graph Problem:

  $PATH = \{\langle G, s, t\rangle |\ G$ is a directed graph that has a directed path from s to $t\}$

- ☑ • A Number Problem:

  $RELPRIME = \{\langle x, y\rangle |\ x$ and y are relatively prime$\}$

- ☑ • A CFL Problem:

  Every context-free language is a member of P

Check-in Quiz 4/20

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