CS420

NFA -> DFA

Wednesday, February 15, 2023

UMass Boston CS

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

$\Rightarrow$

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

Announcements

- HW 2 in
  - ~~Due Tue 2/14 11:59pm~~
- HW 3 out
  - Due Sun 2/26 11:59pm
  - Note: extended due date
- Office Hours
  - Woody's time moved: Tue 4-5:30pm, McCormack 3rd floor, room 139
- No lecture next Monday 2/20

Quiz Preview

- An "if and only if" statement represents two of what kind of statements?

Concatenation: $A \circ B = \{xy|\, x \in A \text{ and } y \in B\}$

Last Time: Concatenation is Closed?

THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof: Construct a new machine?

Concatenation: $A \circ B = \{xy | \ x \in A \text{ and } y \in B\}$

Concatenation Examples

THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- If: $a_1 \in A_1, a_2 \in A_2$
- If: $a_3 \in A_1, a_4 \notin A_2$
- If: $a_5 \notin A_1, a_6 \in A_2$
- If: $a_7 \notin A_1, a_8 \notin A_2$

- Then: $a_1 a_2 \in A_1 \circ A_2$???
- Then: $a_3 a_4 \in A_1 \circ A_2$???
- Then: $a_5 a_6 \in A_1 \circ A_2$???
- Then: $a_7 a_8 \in A_1 \circ A_2$???

Then, in proof, explain why the constructed machine accepts this string

Concatenation: $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$

Last Time: Concatenation is Closed?

THEOREM

The class of regular languages is closed under the concatenation operation.

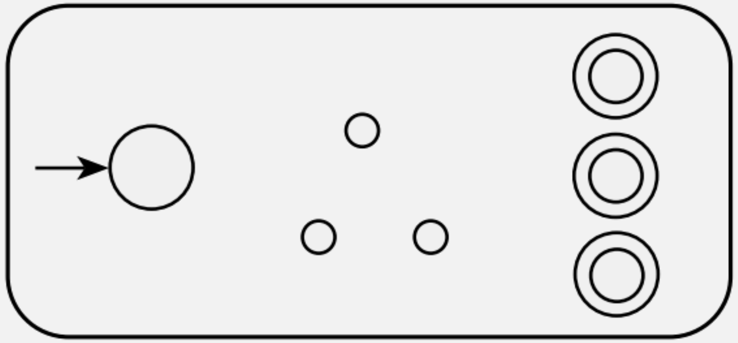
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof: Construct a new machine?

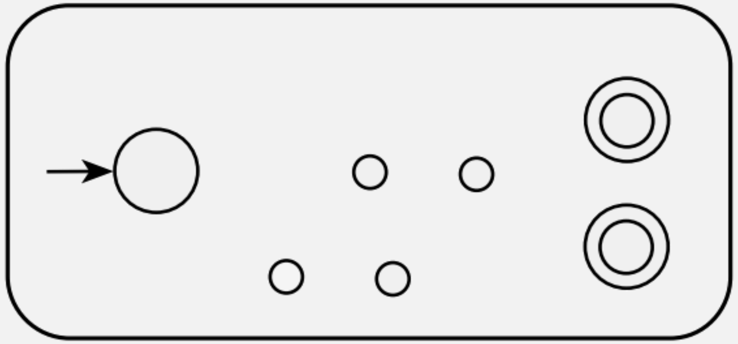
- How does it know when to switch machines?
  - Can only read input once

Concatentation

M_1



M_2



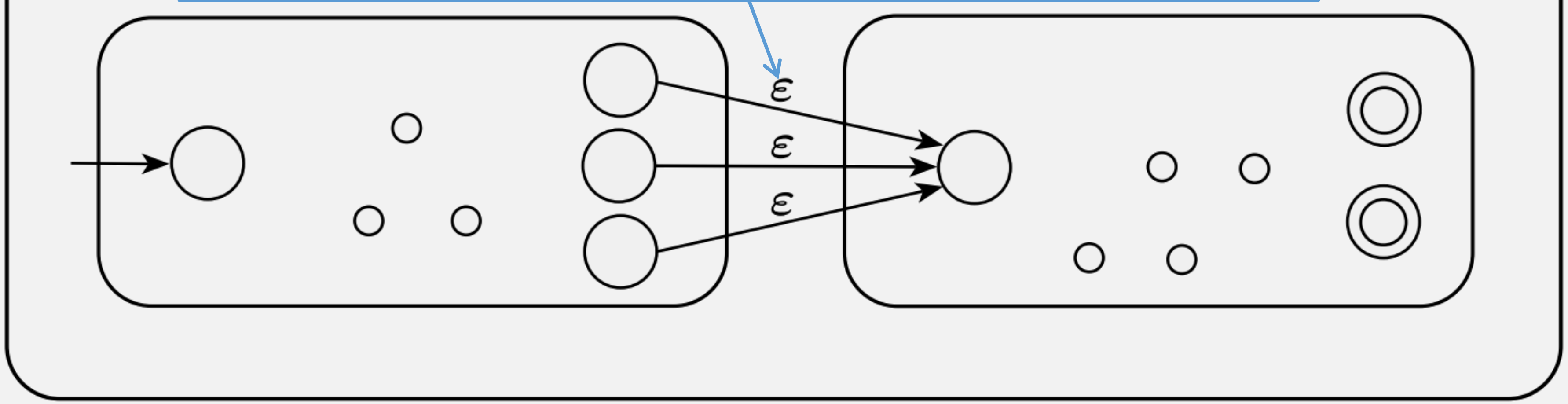
Let M_1 recognize A_1, and M_2 recognize A_2.

Want: Construction of N to recognize $A_1 \circ A_2$

N is an **NFA**! It can:
- Keep checking 1st part with M_1
  and
- Move to M_2 to check 2nd part

N

ε = "empty transition" = reads no input
Allows N to be in both machines at the same time!



Concatenation is Closed for Regular Langs

PROOF

Let DFA $M_1 = (\boxed{Q_1}, \Sigma, \delta_1, \boxed{q_1}, F_1)$ recognize A_1

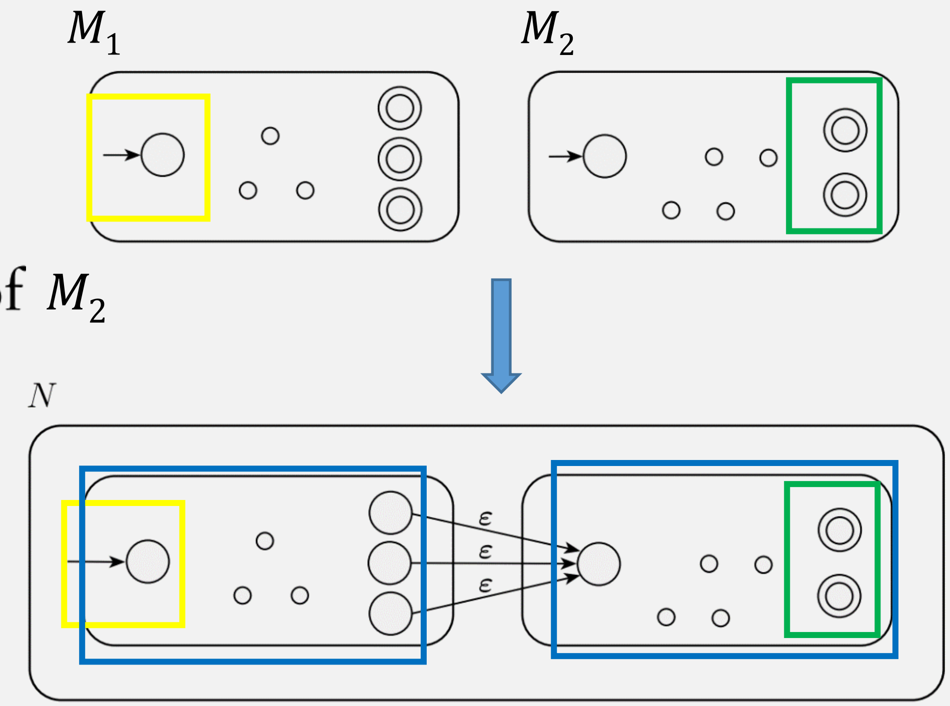
DFA $M_2 = (\boxed{Q_2}, \Sigma, \delta_2, q_2, \boxed{F_2})$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $\boxed{Q} = Q_1 \cup Q_2$

2. The state $\boxed{q_1}$ is the same as the start state of M_1

3. The accept states $\boxed{F_2}$ are the same as the accept states of M_2



Concatenation is Closed for Regular Langs

Wait, is this true?

PROOF

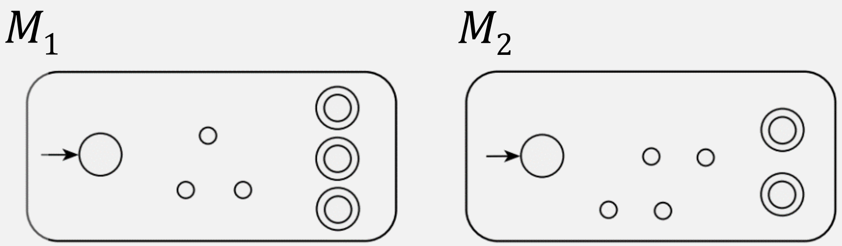
Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

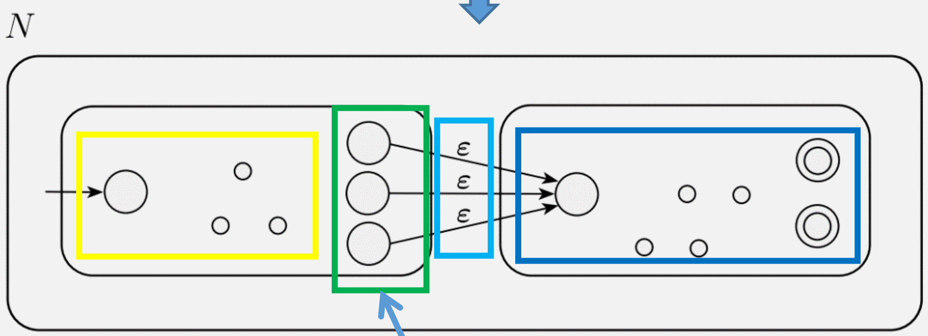
NFA def says δ must map every state and ε to some set of states

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) \; ? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) \; ? & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) \; ? & q \in Q_2. \end{cases}$$





And: $\delta(q, \varepsilon) = \emptyset$, for $q \in Q$, $q \notin F_1$??? $\blacksquare$

Flashback: Is Union Closed For Regular Langs?

Statements	Justifications
1. A_1 and A_2 are regular languages	1. Assumption
2. A DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes A_1	2. Def of Regular Language
3. A DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes A_2	3. Def of Regular Language
4. Construct DFA $M = (Q, \Sigma, \delta, q_0, F)$	4. Def of DFA
5. M recognizes $A_1 \cup A_2$	5. See examples
6. $A_1 \cup A_2$ is a regular language	6. Def of Regular Language
7. The class of regular languages is closed under the union operation. In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.	7. From stmt #1 and #6

Is <u>Concat</u> Closed For Regular Langs?

Statements

1. A_1 and A_2 are regular languages
2. A DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes A_1
3. A DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes A_2
4. Construct **NFA** $N =$ **???** (todo)
5. N recognizes $\cancel{A_1 \cup A_2}$ $\boxed{A_1 \circ A_2}$
6. $\boxed{A_1 \circ A_2}$ $\cancel{A_1 \cup A_2}$ is a regular language
7. The class of regular languages is closed under the concatenation operation.

   In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Justifications

1. Assumption
2. Def of Regular Language
3. Def of Regular Language
4. Def of **NFA**
5. See examples
6. Does NFA recognize regular lang**?**
7. From stmt #1 and #6

Flashback: A DFA's Language

- For **DFA** $M = (Q, \Sigma, \delta, q_0, F)$

- *M* **accepts** *w* if $\hat{\delta}(q_0, w) \in F$

- *M* **recognizes** language $\{w \mid M \text{ accepts } w\}$

Definition: A DFA's language is a **regular language**

An NFA's Language

- For **NFA** $N = (Q, \Sigma, \delta, q_0, F)$

intersection

accept states

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$ ← not empty
  - i.e., accept if final states contain <u>at least one</u> accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

<u>Q</u>: What kind of languages do NFAs recognize?

Concatenation Closed for Reg Langs?

- **Combining DFAs to recognize concatenation of languages …**

  … produces an <u>NFA</u>

- So **to prove concatenation is closed …**

  … we must prove that <u>NFAs *also* recognize regular languages</u>.

Specifically, we must <u>prove</u>:

NFAs ⇔ regular languages

“If and only if” Statements

$X \Leftrightarrow Y$ = “X if and only if Y” = X iff Y = X <=> Y

Represents two statements:

1. $\Rightarrow$ if X, then Y
   - “**forward**” direction

2. $\Leftarrow$ if Y, then X
   - “**reverse**” direction

How to Prove an "iff" Statement

$X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y

Proof has <u>two</u> (If-Then proof) parts:

1. $\Rightarrow$ if X, then Y
   - "**forward**" direction
   - assume X, then use it to prove Y
2. $\Leftarrow$ if Y, then X
   - "**reverse**" direction
   - assume Y, then use it to prove X

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

$\Rightarrow$ If L is regular, then some NFA N recognizes it.

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 2)

$\Leftarrow$ If an NFA N recognizes L, then L is regular.

Statements & Justifications?

"equivalent" = "recognizes the same language"

⇒ If L is regular, then some NFA N recognizes it

Statements	**Justifications**
1. L is a regular language	1. Assumption
2. A DFA M recognizes L	2. Def of Regular language
3. Construct NFA N equiv to M	3. See hw 2!
4. An NFA N recognizes L	4. ???
5. If L is a regular language, then some NFA N recognizes it	5. By Stmts #1 and #4

Assume the "if" part ...

... use it to prove "then" part

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

☑ ⇒ If L is regular, then some NFA N recognizes it.

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- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 2)

⇐ If an NFA N recognizes L, then L is regular.

(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA N → an equivalent DFA

"equivalent" =
"recognizes the same language"

How to convert NFA→DFA?

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
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A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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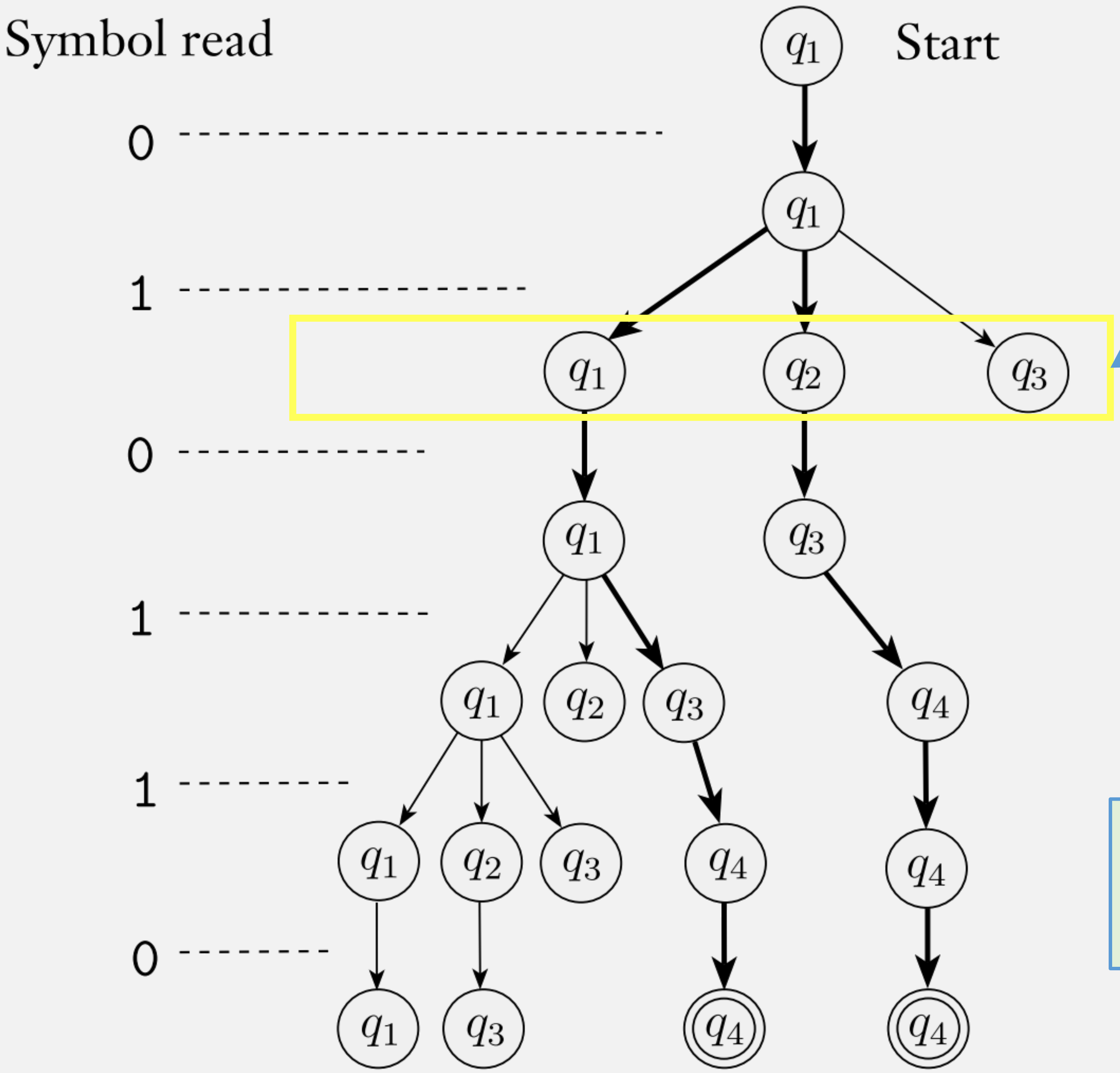
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA
= set of states in the NFA



NFA computation can be in multiple states

DFA computation can only be in one state

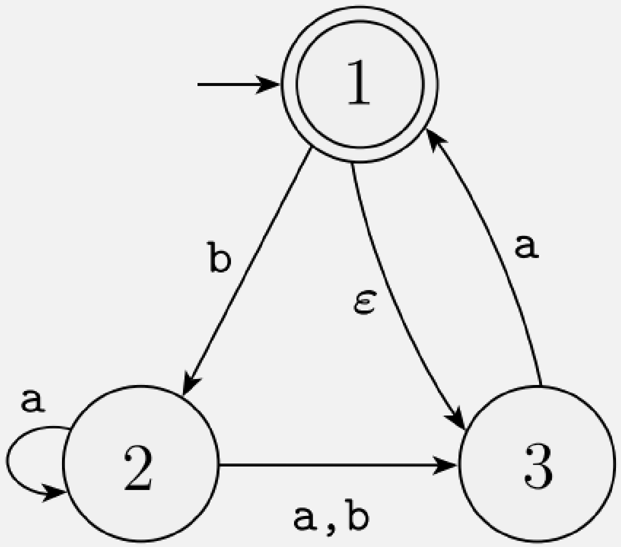
So **encode**: a set of NFA states as one DFA state

This is similar to the proof strategy from "**Closure of union**" where: **a state = a pair of states**

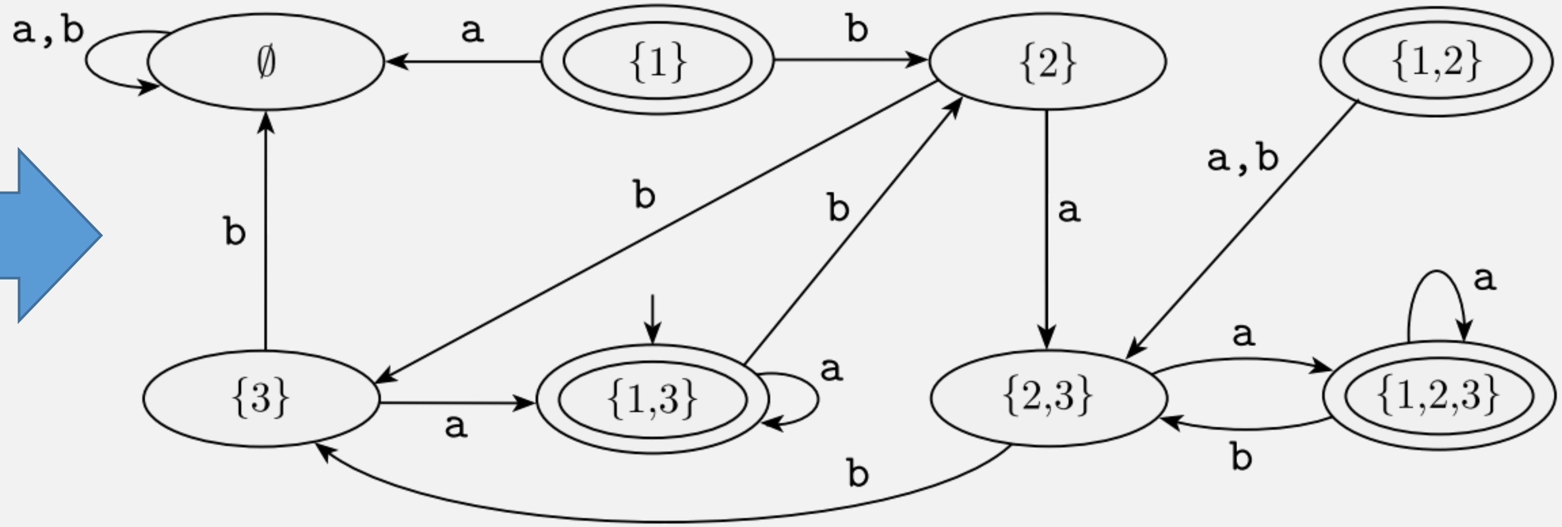
Convert **NFA→DFA**, Formally

- Let **NFA** N = $(Q, \Sigma, \delta, q_0, F)$
- **An equivalent** DFA M **has states** Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Have: **NFA** $N = (Q, \Sigma, \delta, q_0, F)$

Want: **DFA** $M = (Q', \Sigma, \delta', q_0', F')$

$\mathbf{1.}\ Q' = \mathcal{P}(Q)$

A DFA state = a set of NFA states

$\mathbf{2.}$ For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

A DFA step = an NFA step for all states in the set

R = DFA state = set of NFA states

$\mathbf{3.}\ q_0' = \{q_0\}$

$\mathbf{4.}\ F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Flashback: Adding Empty Transitions

- Define the set $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

- Recursive case:

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA

Have: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Want: DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

1. $Q' = \mathcal{P}(Q)$

Almost the same, except …

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \cancel{\delta(r, a)} \ \varepsilon\text{-REACHABLE}(\delta(r, a))$$

But this produces a set!
We need another
"reachable" function
(see hw 3!)

3. ${q_0}' = \cancel{\{q_0\}} \ \varepsilon\text{-REACHABLE}(q_0)$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Proving NFAs Recognize Regular Langs

Theorem:
A language L is regular **if and only if** some NFA N recognizes L.

Proof:
$\Rightarrow$ If L is regular, then some NFA N recognizes it.
(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA $\rightarrow$ an equivalent NFA! (see HW 2)

$\Leftarrow$ If an NFA N recognizes L, then L is regular.
(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA $N \rightarrow$ an equivalent DFA …
… using our NFA to DFA algorithm! $\blacksquare$

Concatenation is Closed for $\boxed{\text{Regular Langs}}$

Wait, is this true?

PROOF

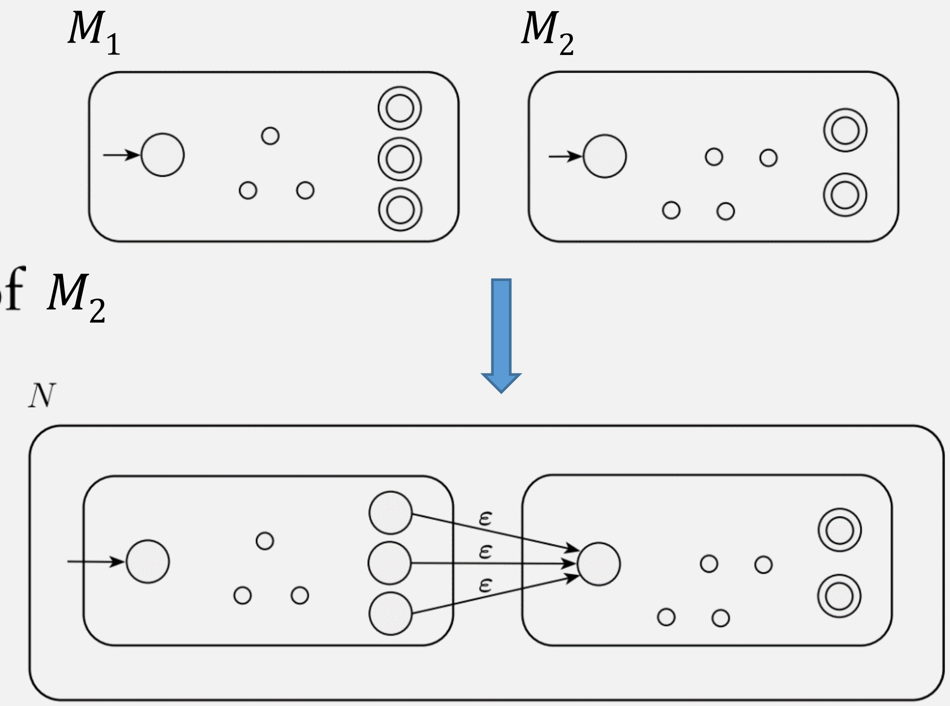
Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

If a language has an **NFA** recognizing it, then **it is a regular** language

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$



☑ ~~???~~ ■

Concat Closed for Reg Langs: Use NFAs Only

PROOF

If language is **regular**,
then **it has an NFA recognizing it** …

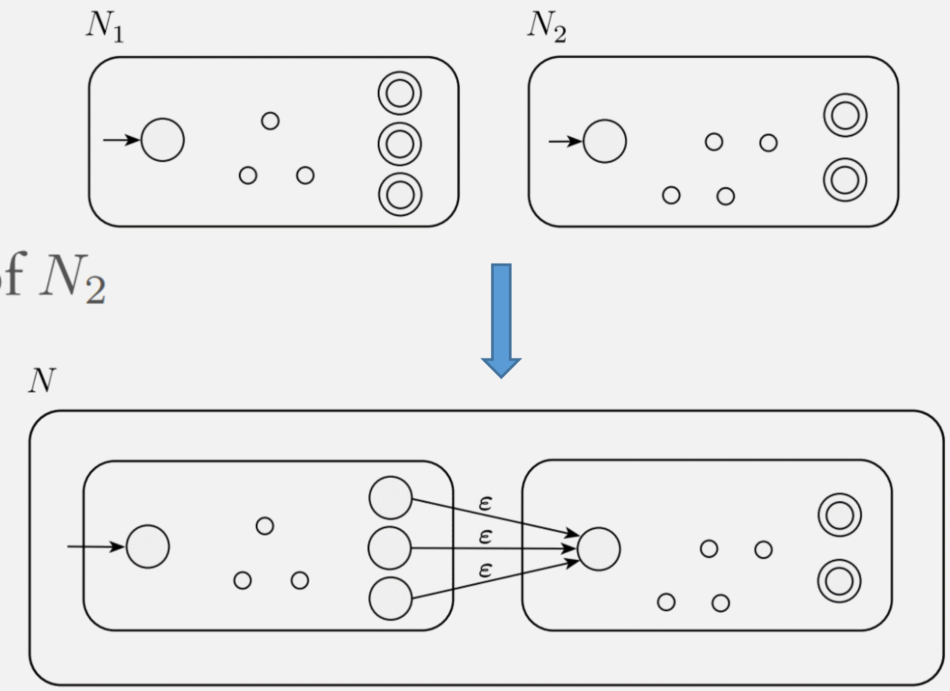
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

????

$$\delta(q, a) = \begin{cases} ? \quad \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ ? \quad \delta_1(q, a) & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ ? \quad \delta_2(q, a) & q \in Q_2. \end{cases}$$



Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Flashback: Union is Closed For Regular Langs

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Combine the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Flashback: Union is Closed For Regular Langs

Proof

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: a <u>new</u> machine $M = (Q, \Sigma, \delta, q_0, F)$ using M_1 and M_2
- states of M: $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2

State in M = M_1 state + M_2 state

- M transition fn: $\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big)$

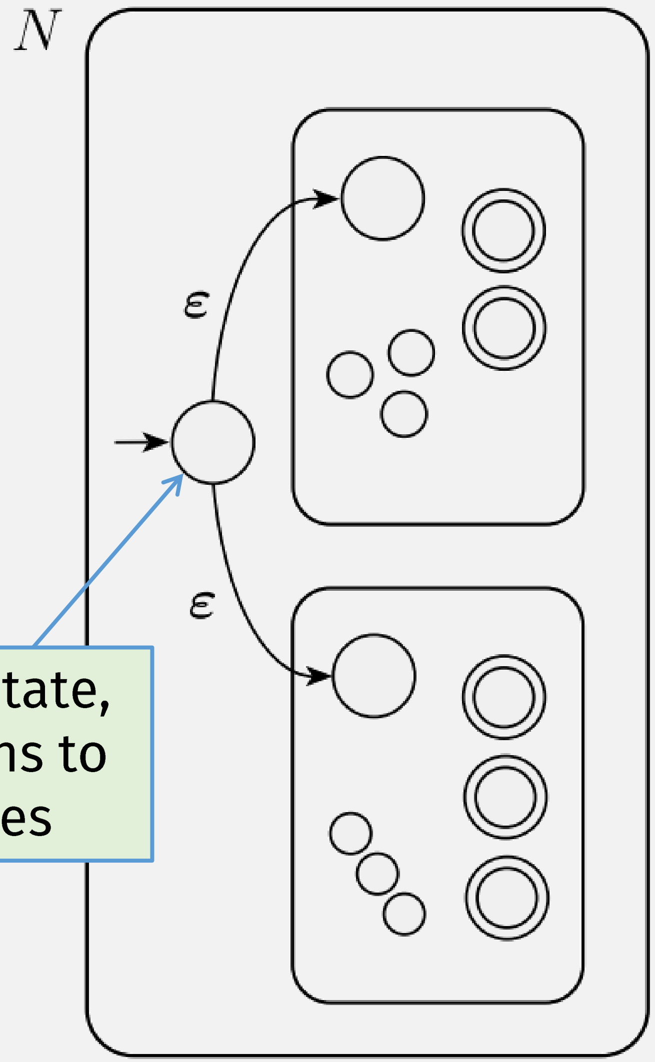
M step = a step in M_1 + a step in M_2

- M start state: (q_1, q_2)

Accept if <u>either</u> M_1 or M_2 accept

- M accept states: $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

Alternate Proof, with NFAs

Union is Closed for Regular Languages

PROOF

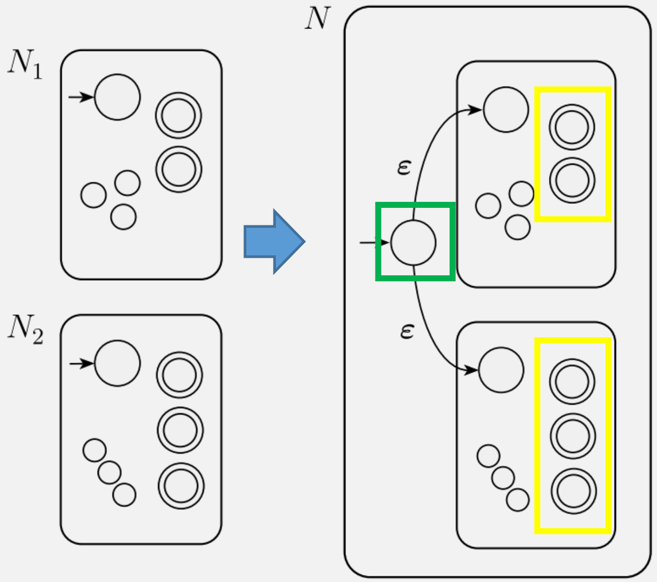
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, \boxed{q_0,} F)$ to recognize $A_1 \cup A_2$.

1. $Q = \boxed{\{q_0\}} \cup Q_1 \cup Q_2$.

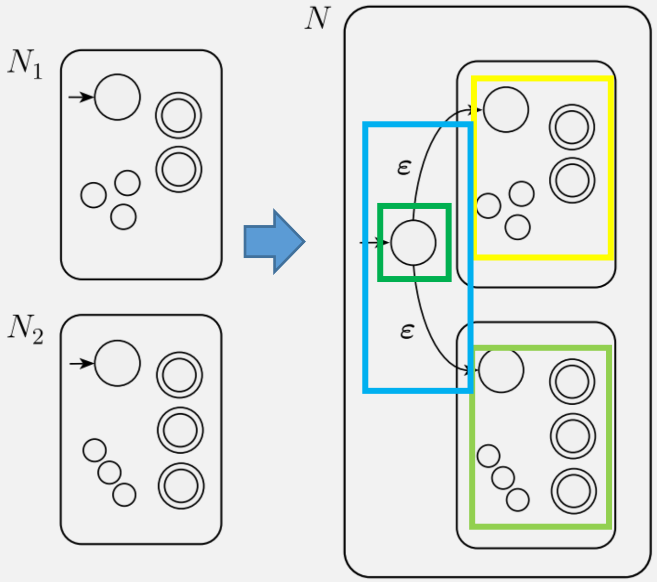
2. The state $\boxed{q_0}$ is the start state of N.

3. The set of accept states $\boxed{F} = F_1 \cup F_2$.



Alternate Proof, with NFAs

Union is Closed for Regular Languages



PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.

2. The state q_0 is the start state of N.

3. The set of accept states $F = F_1 \cup F_2$.

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(?, a) & \boxed{q \in Q_1} \\ \delta_2(?, a) & \boxed{q \in Q_2} \\ \{q_1 ? q_2\} & \boxed{q = q_0 \text{ and } a = \varepsilon} \\ \emptyset \quad ? & \boxed{q = q_0 \text{ and } a \neq \varepsilon} \end{cases}$$

Don't forget Statements and Justifications!

List of Closed Ops for Reg Langs (so far)

- ☑ • Union
- ☑ • Concatentation
- • Kleene Star (repetition) ?

Star: $A^* = \{x_1x_2\ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

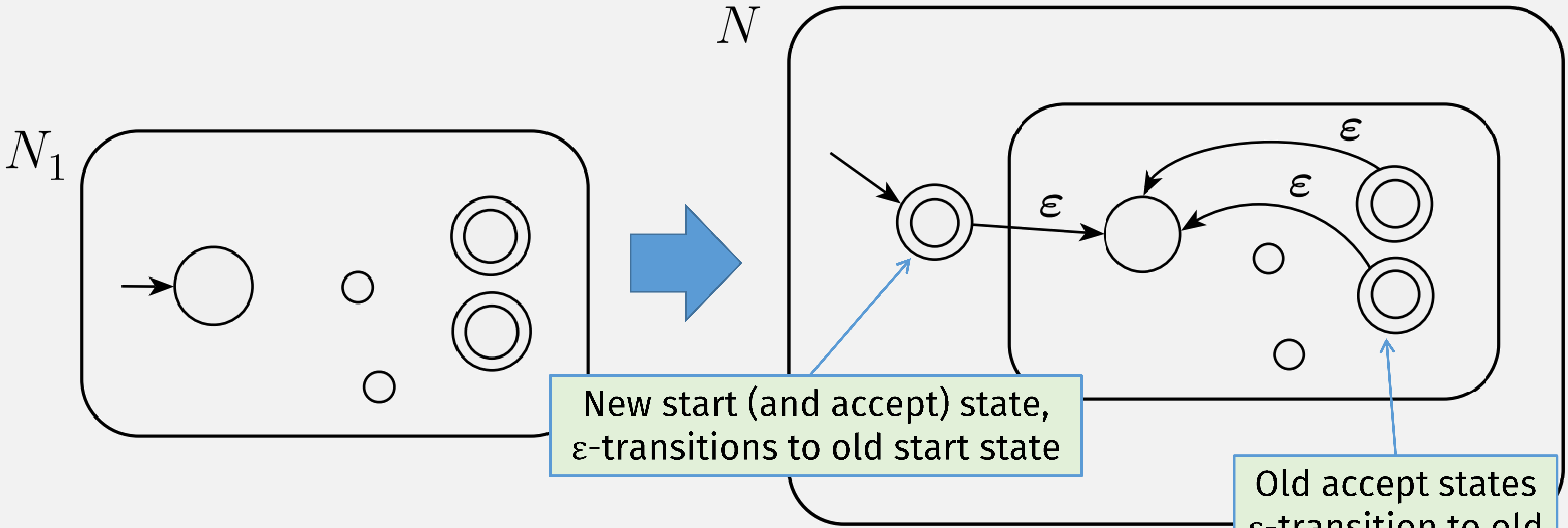
If $A = \{\mathtt{good}, \mathtt{bad}\}$

$$A^* = \{\varepsilon, \mathtt{good}, \mathtt{bad}, \mathtt{goodgood}, \mathtt{goodbad}, \mathtt{badgood}, \mathtt{badbad}, \mathtt{goodgoodgood}, \mathtt{goodgoodbad}, \mathtt{goodbadgood}, \mathtt{goodbadbad}, \ldots\}$$

Note: repeat zero or more times

(this is an infinite language!)

Kleene Star



New start (and accept) state, ε-transitions to old start state

Old accept states ε-transition to old start state

In-class exercise:

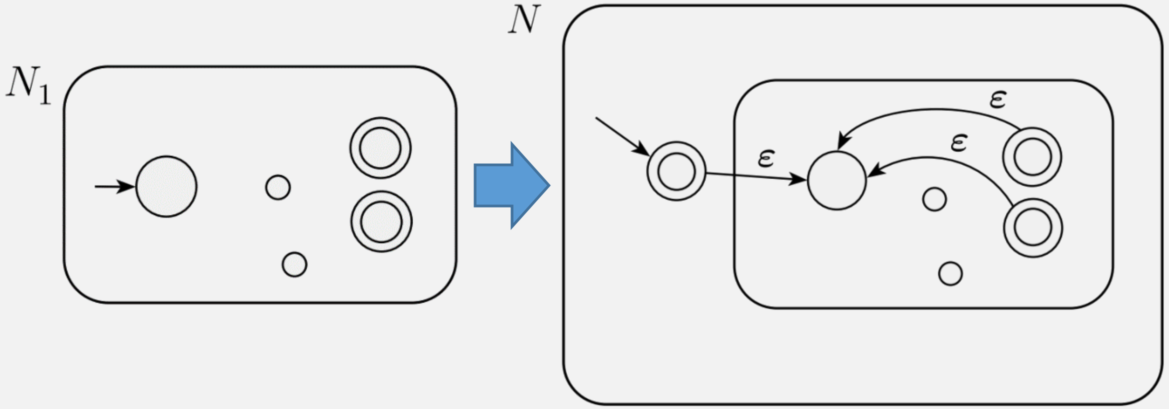
Kleene Star is Closed for Regular Langs

THEOREM

The class of regular languages is closed under the star operation.

Kleene Star is Closed for Regular Langs

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.



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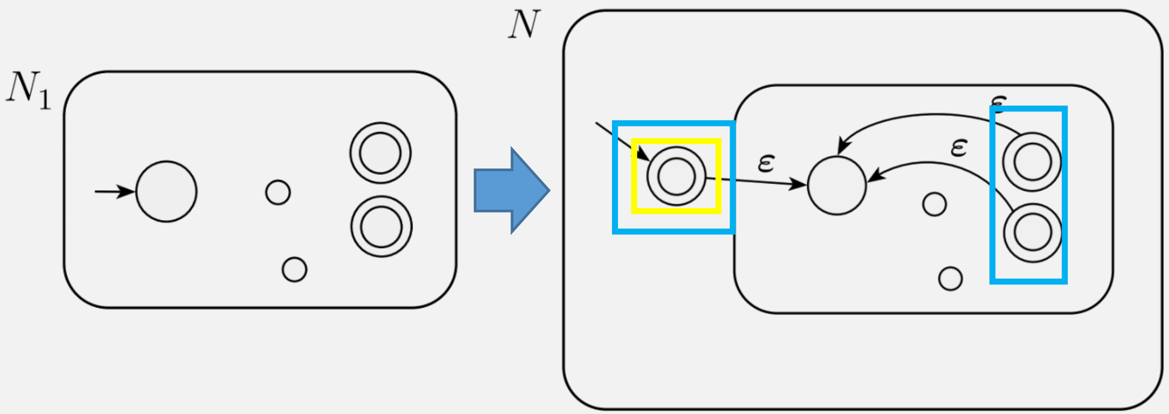

1. $Q = \boxed{\{q_0\}} \cup Q_1$
2. The state $\boxed{q_0}$ is the new start state.
3. $F = \boxed{\{q_0\} \cup F_1}$

Kleene star of a language must accept the empty string!

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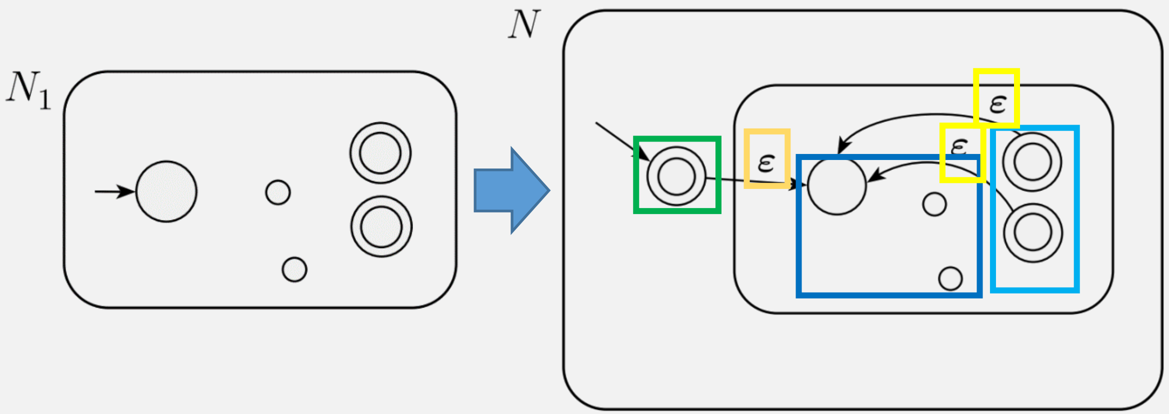

1. $Q = \{q_0\} \cup Q_1$

2. The state q_0 is the new start state.

3. $F = \{q_0\} \cup F_1$

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a)\,? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a)\,? & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a)\,? \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} \quad ? & q = q_0 \text{ and } a = \varepsilon \\ \emptyset \quad ? & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Next Time: Why These Closed Operations?

- Union
- Concat
- Kleene star

<u>All</u> regular languages can be <u>constructed</u> from:
- single-char strings, and
- these three combining operations!

Check-in Quiz 2/15

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