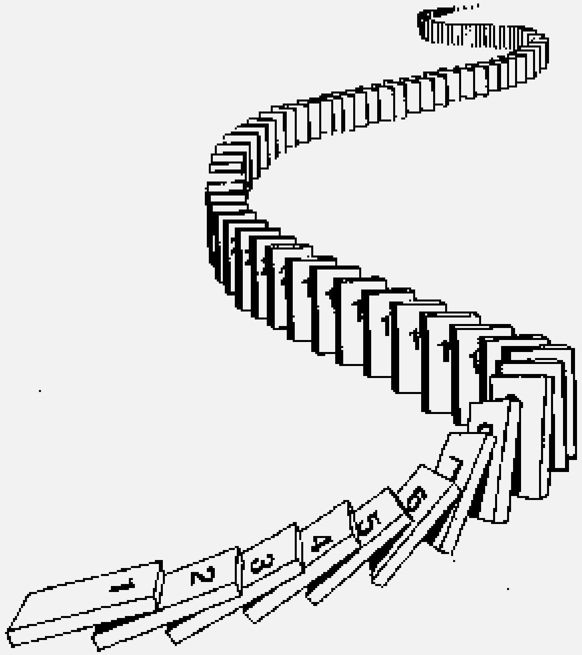
UMB CS 420

Inductive Proofs

Monday February 27, 2023



Announcements

- HW 3 in
  - ~~Due Sun 2/26 11:59pm EST~~

- HW 4 out
  - Due Sun 3/5 11:59pm EST

Quiz Preview

- Which of the following best describes when to use a **proof by induction**?

Kinds of Mathematical Proof

- **Deductive proof** (from before)
  - Starting from assumptions and known definitions,
  - Reach conclusion by making logical inferences

- **Proof by induction** (now)
  - …
  - Use this when working with <u>recursive</u> definitions

A <u>valid</u> <u>recursive definition</u> has:
- **base case**(s) and
- **recursive case**(s) (with "smaller" self-reference)

Proof by Induction

To Prove: a ***Statement*** about a recursively defined "thing" x:

1. Prove: *Statement* for base case of x
2. Prove: *Statement* for recursive case of x:
   - Assume: **induction hypothesis** (IH)

     I.e., *Statement* is true for some $x_{smaller}$
     - E.g., if x is number, then "smaller" = lesser number

   - Prove: *Statement* for x_{larger}, using IH (and known definitions, theorems …)
     - Typically: show that going from $x_{smaller}$ to x_{larger} preserves *Statement*

A valid recursive definition has:
- **base case**(s) and
- **recursive case**(s) (with "smaller" self-reference)

Natural Numbers Are Recursively Defined

A **Natural Number** is:

Base Case
- **0**

Recursive Case
- Or $\boldsymbol{k+1}$, where k is a Natural Number

Self-reference

But definition is valid because self-reference is "smaller"

So proving things about Natural Numbers requires **proof by induction**!

A valid recursive definition has:
- **base case** and
- **recursive case** (with "smaller" self-reference)

Proof By Induction Example (Sipser Ch 0)

Prove true: $$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

- P_t = loan balance after t months
- t = # months
- P = principal = original amount of loan
- M = interest (multiplier)
- Y = monthly payment

(Details of these variables not too important here)

Proof By Induction Example (Sipser Ch 0)

Prove true: $P_t = PM^{t} - Y\left(\frac{M^{t}-1}{M-1}\right)$

Proof: by **induction** on natural number t

An proof by induction exactly follows the recursive definition (here, natural numbers) that the induction is "on"

Base Case, $t = 0$:

- Goal: **Show** $P_0 = P$ (amount owed at start = loan amount)
- Proof of Goal:

$$P_0 = PM^0 - Y\left(\frac{M^0-1}{M-1}\right) = P$$

Plug in $t = 0$

Simplify, to get to goal statement

A Natural Number is:

- 0
- Or $k + 1$, where k is a natural number

Proof By Induction Example (Sipser Ch 0)

A **proof by induction** exactly follows the **recursive definition** (here, natural numbers) **that the induction is "on"**

A Natural Number is:

- 0
- k + 1, for some nat num k

Prove true: $$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

Inductive Case: $t = k + 1$, for some nat num k

- Inductive Hypothesis (IH), **assume statement true for some** t = (smaller) k

"Connect together" known definitions and statements

$$P_k = PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)$$

- Goal **statement to prove, for** $t = k+1$:

$$P_{k+1} = PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

Plug in IH

Simplify, to get to goal statement

- Proof of Goal:

$$P_{k+1} = P_k M - Y$$

Definition of P_{k+1}

In-class Exercise: Proof By Induction

A **proof by induction** exactly follows the recursive definition (here, natural numbers) that the induction is "on"

A Natural Number is:

- 0
- $k + 1$, for some nat num k

<u>Prove</u>: $(z \neq 1)$

$$\sum_{i=0}^{m} z^i = \frac{1 - z^{m+1}}{1 - z}$$

Use **Proof by Induction.**

Make sure to clearly state what (number) the induction is "on"

Proof by Induction: CS 420 Example

Statement to prove: LANGOF (G) = LANGOF (R = **GNFA→RegExpr**(G))

Condition for **GNFA→RegExpr** function to be "correct", i.e., the languages must be equivalent

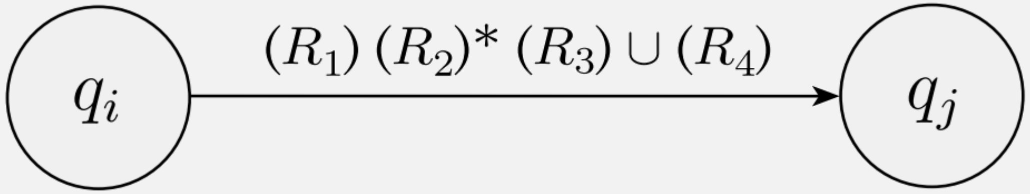
- Where:
  - G = a GNFA
  - R = a Regular Expression
  - R = **GNFA→RegExpr**(G)
- i.e., **GNFA→RegExpr** must not change the language!
  - Key step: the rip/repair step

Last Time: GNFA→RegExpr (recursive) function

On GNFA input G:

Base Case

- If G has 2 states, **return** the regular expression (from the transition), e.g.:



Recursive Case

- Else:
  - "Rip out" one state
  - "Repair" the machine to get an equivalent GNFA G'
  - Recursively call GNFA→RegExpr(G')

Recursive definitions have:
- base case and
- recursive case
  (with a "smaller" object)

Proof by Induction: CS 420 Example

Statement to prove: LANGOF (G) = LANGOF (**GNFA→RegExpr**(G))

Recursively defined "thing"

Plug in

Proof: by Induction on # of states in G

☑ 1. Prove *Statement* is true for base case

G has 2 states $\quad (q_i) \xrightarrow{R} (q_j)$

Why is this an ok base case?

Statements	**Justifications**
1. LANGOF ($(q_i) \xrightarrow{R} (q_j)$) = LANGOF (R)	1. Definition of GNFA
2. GNFA→RegExpr($(q_i) \xrightarrow{R} (q_j)$) = R	2. Definition of GNFA→RegExpr
LANGOF ($(q_i) \xrightarrow{R} (q_j)$) = LANGOF (GNFA→RegExpr($(q_i) \xrightarrow{R} (q_j)$))	3. From (1) and (2)

Goal

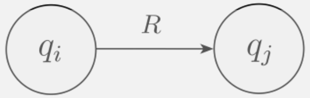
Don't forget to write out Statements / Justifications !

Proof by Induction: CS 420 Example

Statement to prove: LANGOF (G) = LANGOF (**GNFA→RegExpr**(G))

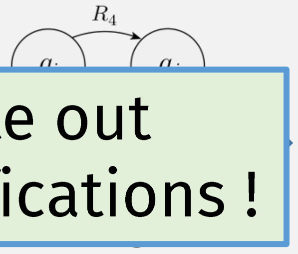
Proof: by Induction on # of states in G

☑ 1. Prove *Statement* is true for base case — G has 2 states



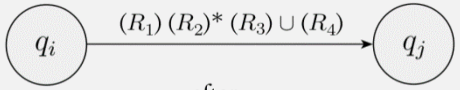
2. Prove *Statement* is true for recursive case: — G has > 2 states
   - Assume the **induction hypothesis** (IH):
     - *Statement* is true for smaller G'
   - Use it to prove *Statement* is true for larger G
     - Show that going from G to G' preserves *Statement*

LANGOF (G')
=
LANGOF (**GNFA→RegExpr**(G'))
(Where G' has less states than G)



before

Don't forget to write out Statements / Justifications !



after

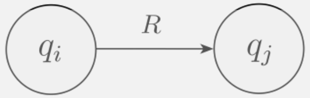
Show that "rip/repair" step converts G to smaller, equivalent G'

Proof by Induction: CS 420 Example

Statement to prove: **LANGOF (*G*) = LANGOF (GNFA→RegExpr(*G*))**

Proof: by Induction on # of states in *G*

☑ 1. Prove *Statement* is true for base case — *G* has 2 states



☑ 2. Prove *Statement* is true for recursive case: — *G* has > 2 states

- Assume the **induction hypothesis** (IH):
  - *Statement* is true for smaller *G'*
- Use it to prove *Statement* is true for larger *G*
  - Show that going from *G* to *G'* preserves *Statement*

LANGOF (*G'*)
=
LANGOF (GNFA→RegExpr(*G'*))
(Where *G'* has less states than *G*)

Statements	**Justifications**
1. LANGOF (*G'*) = LANGOF (GNFA→RegExpr(*G'*))	1. IH
2. LANGOF (*G*) = LANGOF (*G'*)	2. Correctness of Rip/Repair step (prev)
3. GNFA→RegExpr(*G*) = GNFA→RegExpr(*G'*)	3. Def of GNFA→RegExpr
Goal 4. LANGOF (*G*) = LANGOF (GNFA→RegExpr(*G*))	4. From (1), (2), and (3)

So Far: How to Prove A Language Is Regular?

- Construct DFA
- Construct NFA
- Create Regular Expression ← Slightly different because of recursive definition

R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Proof by Induction

To Prove: a ***Statement*** about a recursively defined "thing" x:

1. Prove: *Statement* for base case of x
2. Prove: *Statement* for recursive case of x:
   - Assume: **induction hypothesis** (IH)

     I.e., *Statement* is true for some $x_{smaller}$
     - E.g., if x is number, then "smaller" = lesser number
     - E.g., if x is regular expression, then "smaller" = ...
   - Prove: *Statement* for x_{larger}, using IH (and known definitions, theorems ...)
     - Usually, must show that going from $x_{smaller}$ to x_{larger} preserves *Statement*

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,

“smaller”

Whole reg expr

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Thm: Reverse is Closed for Regular Langs

$\texttt{abc}^{\mathcal{R}} = \texttt{cba}$

For any string $w = w_1 w_2 \cdots w_n$, the ***reverse*** of w, written $w^{\mathcal{R}}$, is the string w in reverse order, $w_n \cdots w_2 w_1$.

For any language A, let $A^{\mathcal{R}} = \{w^{\mathcal{R}} | \, w \in A\}$

$\{\texttt{a}, \texttt{ab}, \texttt{abc}\}^{\mathcal{R}} = \{\texttt{a}, \texttt{ba}, \texttt{cba}\}$

Theorem: if A is regular, so is $A^{\mathcal{R}}$

Proof: by induction on the regular expression of A

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (6 cases)

Base cases

1. a for some a in the alphabet Σ, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
2. ε, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular
3. $\emptyset$, — same reg. expr. represents $A^{\mathcal{R}}$ so it is regular

Inductive cases

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Need to Prove: if A is a regular language, described by reg expr $R_1 \cup R_2$, then $A^{\mathcal{R}}$ is regular

"smaller"

IH1: if A_1 is a regular language, described by reg expr R_1, then ${A_1}^{\mathcal{R}}$ is regular

IH1: if A_2 is a regular language, described by reg expr R_2, then ${A_2}^{\mathcal{R}}$ is regular

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (Case # 4)

Statements	**Justifications**
1. Language A is regular, with reg expr $R_1 \cup R_2$	1. Given
2. R_1 and R_2 are regular expressions	2. Def of Regular Expression
3. R_1 and R_2 describe regular langs A_1 and A_2	3. Reg Expr ⇔ Reg Lang (Prev Thm)
4. If A_1 is a regular language, then $A_1{}^{\mathcal{R}}$ is regular	4. IH
5. If A_2 is a regular language, then $A_2{}^{\mathcal{R}}$ is regular	5. IH
6. $A_1{}^{\mathcal{R}}$ and $A_2{}^{\mathcal{R}}$ are regular	6. By (3), (4), and (5)
7. $A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}}$ is regular	7. Union Closed for Reg Langs
8. $A_1{}^{\mathcal{R}} \cup A_2{}^{\mathcal{R}} = (A_1 \cup A_2)^{\mathcal{R}}$	8. Reverse and Union Ops Commute
9. $A = A_1 \cup A_2$	9. By (1), (2), and (3)
Goal 10. $A^{\mathcal{R}}$ is regular	10. By (7), (8), (9)

Thm: Reverse is Closed for Regular Langs

if A is regular, so is $A^{\mathcal{R}}$

Proof: by Induction on regular expression of A: (6 cases)

Base cases

☑ **1.** a for some a in the alphabet Σ,

☑ **2.** ε,

☑ **3.** $\emptyset$,

Inductive cases

☑ **4.** $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,

5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or

6. (R_1^*), where R_1 is a regular expression.

Remaining cases will use similar reasoning

In-Class quiz 2/27

See gradescope