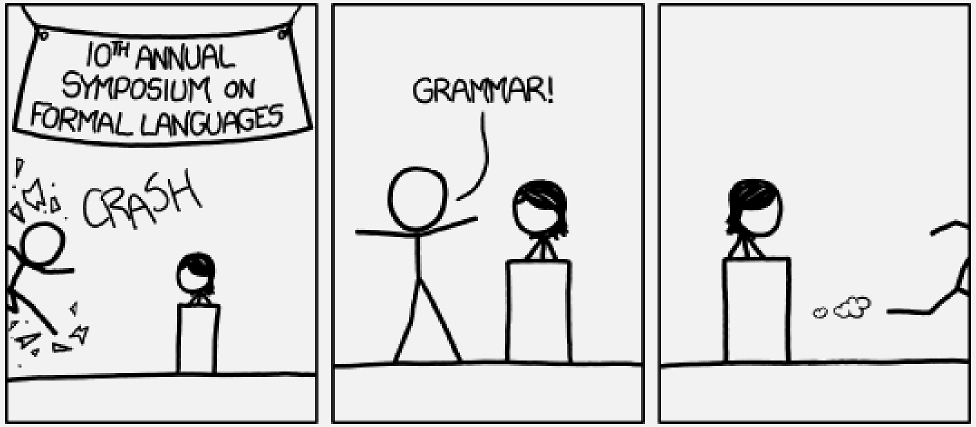
UMB CS 420

Pushdown Automata (PDAs)

Wednesday, March 8, 2023



Announcements

- HW 5 out
  - Due Sun 3/19 11:59pm EST
  - (After Spring Break)

- No lecture next week
  - (Spring Break)

Quiz Preview

1. Which of the following are possible representations of a CFL?
2. Which of the of the following are characteristics of a PDA?
   - Infinite or finite “memory” ?
   - Infinite or finite states ?
   - Deterministic or nondeterministic ?

Last Time: Generating Strings with a CFG

$$G_1 = \begin{array}{l} A \rightarrow 0A1 \\ A \rightarrow B \\ B \rightarrow \# \end{array}$$

A **CFG** represents a **context free language!**

Strings in CFG's language
= all possible generated strings

$$\boxed{L(G_1) \text{ is } \{0^n\#1^n \mid n \geq 0\}}$$

Stop when string is all terminals

A CFG **generates** a string, by repeatedly applying substitution rules:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Start variable

Last Time:

Regular Languages	**Context-Free** Languages (CFLs)
Regular Expression	**Context-Free Grammar** (CFG)
A Reg Expr <u>describes</u> a Regular lang	A **CFG** <u>describes</u> a CFL

Today:

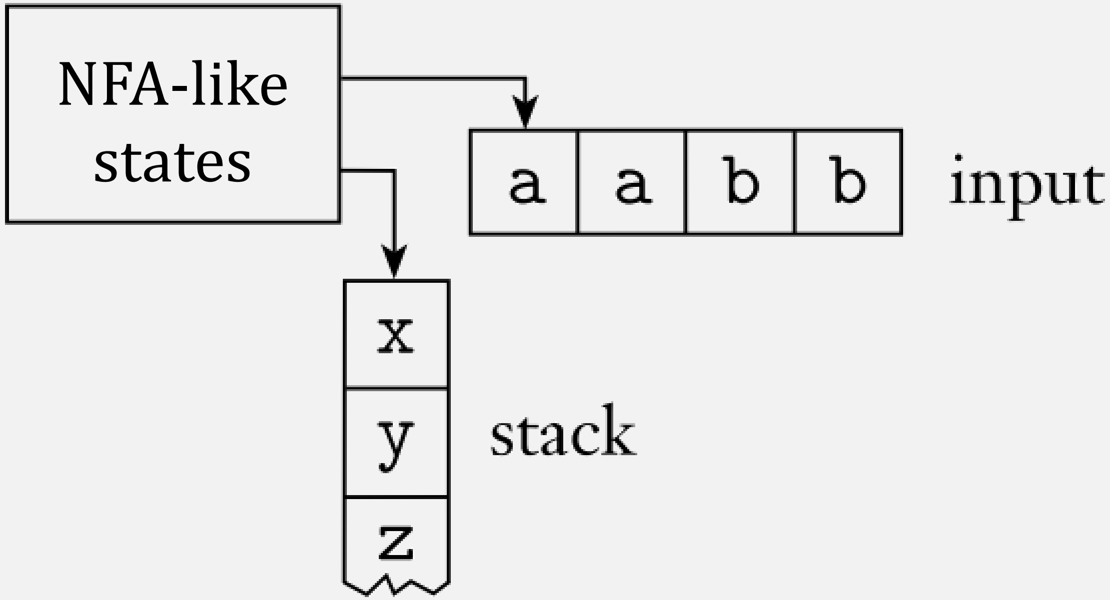
Regular Languages	**Context-Free** Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr describes a Regular lang	A CFG describes a CFL
TODAY:	
Finite Automaton (FSM)	**Push-down automaton** (PDA)
An FSM recognizes a Regular lang	A **PDA** recognizes a CFL

Today:

	Regular Languages	**Context-Free** Languages (CFLs)	
	Regular Expression	Context-Free Grammar (CFG)	
thm	A Reg Expr describes a Regular lang	A CFG describes a CFL	def
	TODAY:		
	Finite Automaton (FSM)	**Push-down automaton** (PDA)	
def	An FSM recognizes a Regular lang	A **PDA** recognizes a CFL	thm
	KEY DIFFERENCE:		
	A Regular lang is defined with a FSM	A CFL is defined with a CFG	
	Must prove: Reg Expr ⇔ Reg lang	*Must prove*: **PDA** ⇔ CFL	

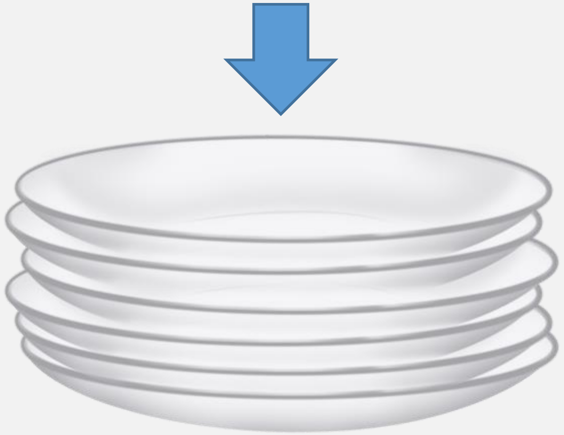
Pushdown Automata (PDA)

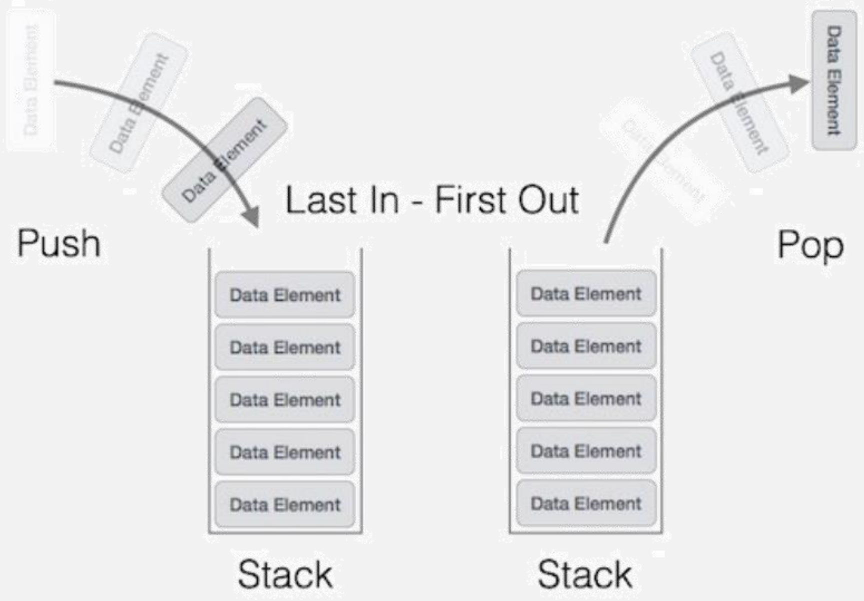
PDA = NFA + a <u>stack</u>



What is a Stack?

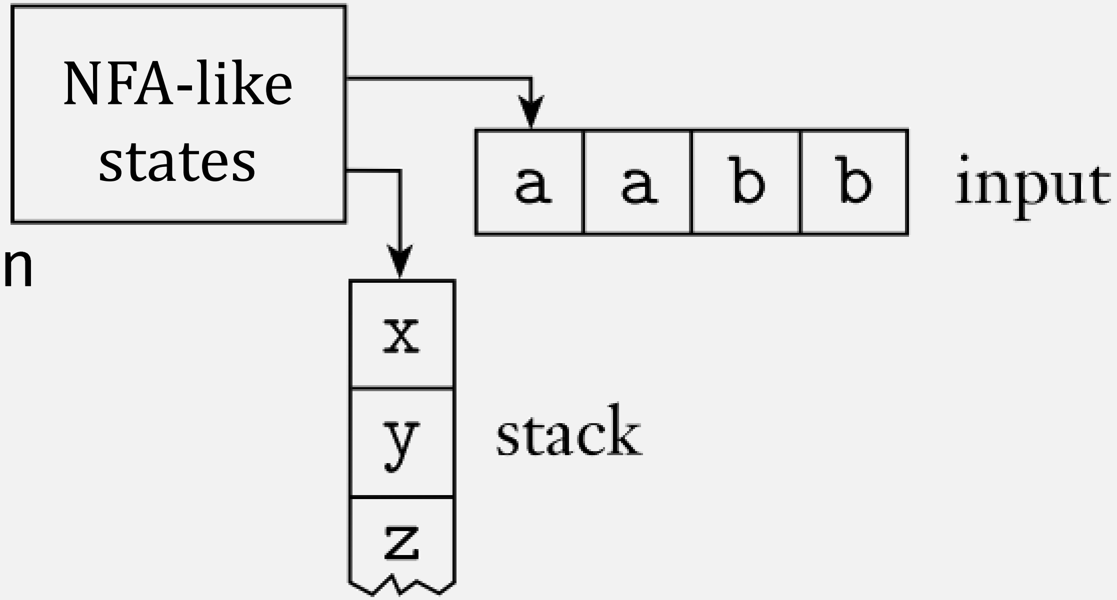
- A <u>restricted</u> kind of (infinite!) memory
- Access to top element only
- 2 Operations only: **push, pop**





Pushdown Automata (PDA)

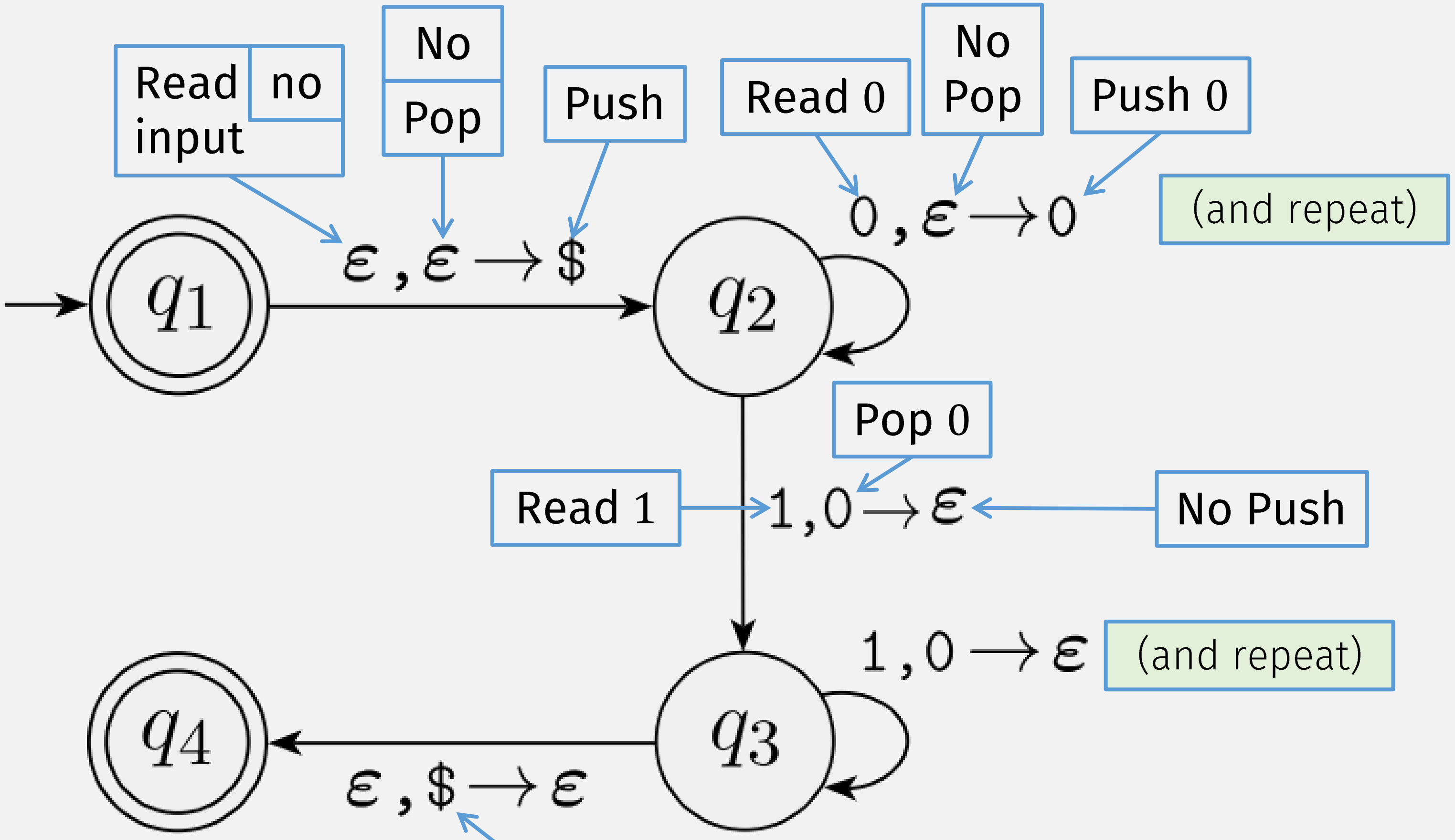
- **PDA** = NFA + a stack
  - Infinite memory
  - Can only read/write top location
    - Push/pop



$\{0^n1^n | \, n \geq 0\}$

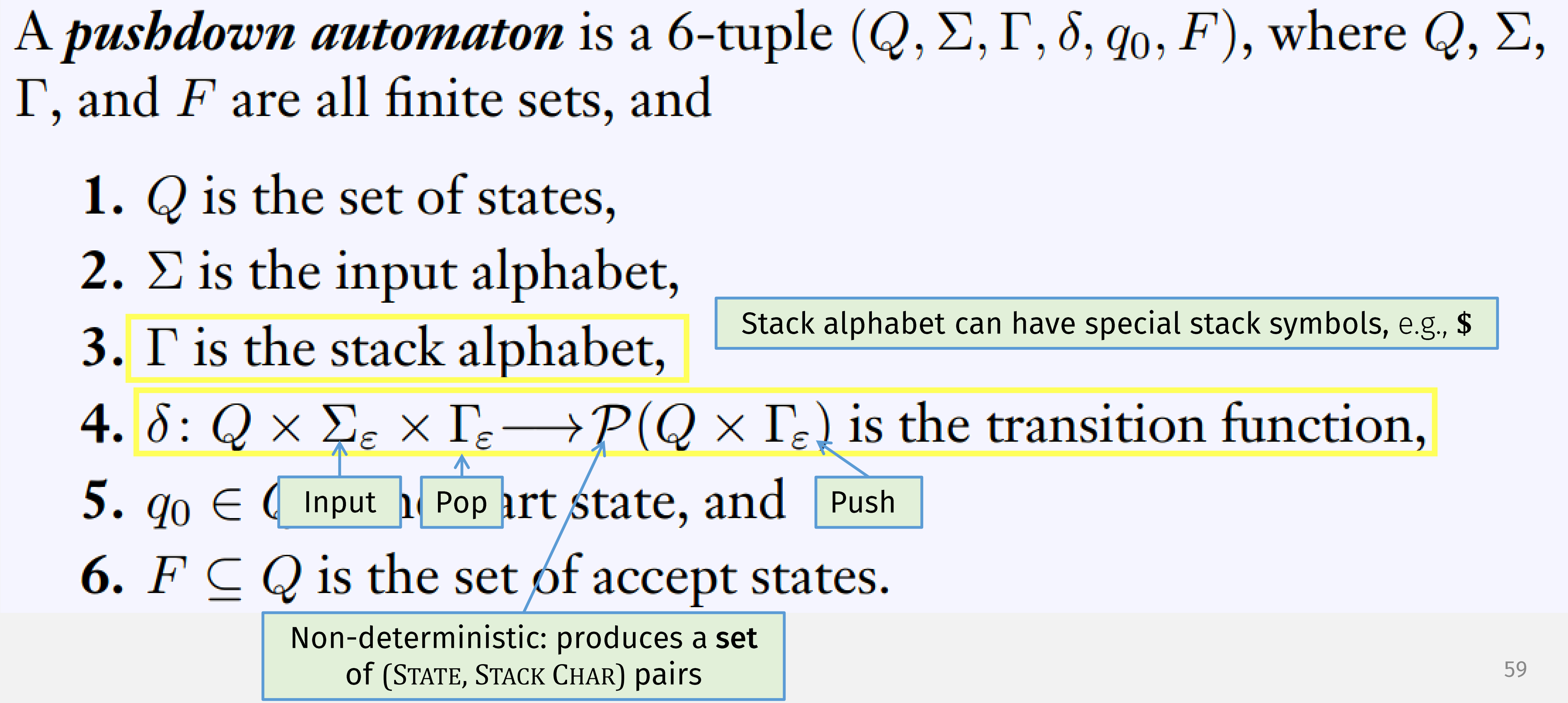
An Example PDA

$ = special symbol, indicating empty stack



Can only pop this (and **accept**) when stack is empty, i.e., when # 0s matches # 1s

Formal Definition of PDA



A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta \colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in$ art state, and
6. $F \subseteq Q$ is the set of accept states.

Stack alphabet can have special stack symbols, e.g., **$**

Input

Pop

Push

Non-deterministic: produces a **set** of (STATE, STACK CHAR) pairs

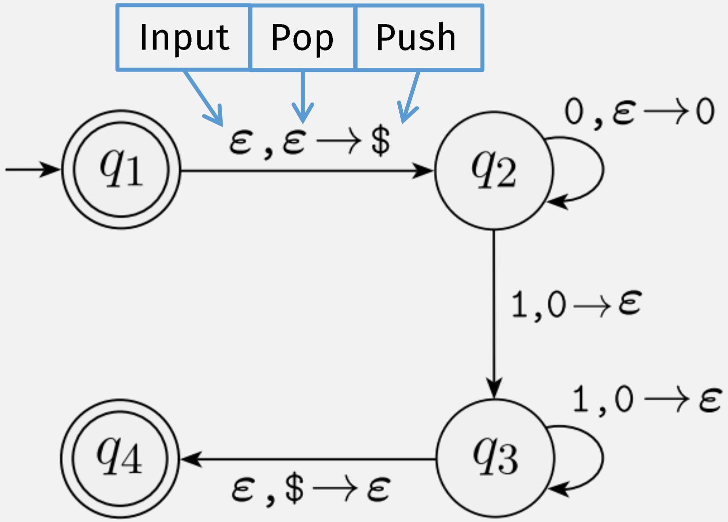
PDA Formal Definition Example

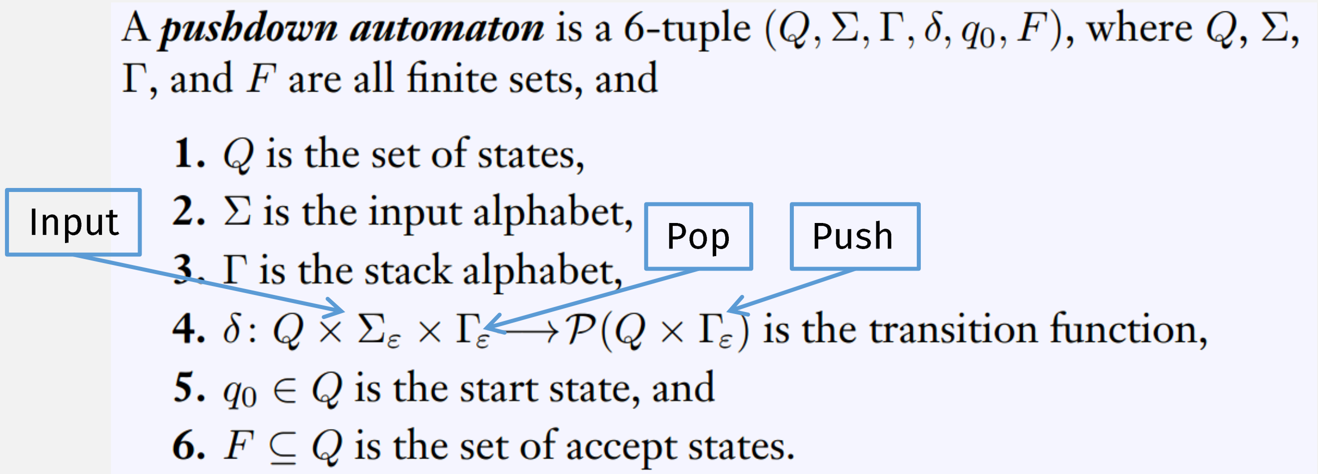
$$Q = \{q_1, q_2, q_3, q_4\},$$

$$\Sigma = \{0,1\},$$

$$\Gamma = \{0, \$\},$$

$$F = \{q_1, q_4\},$$

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$Q = \{q_1, q_2, q_3, q_4\}$,

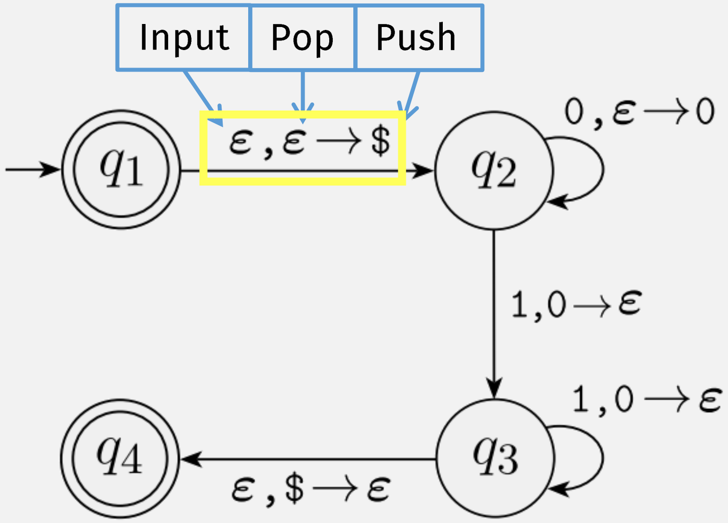
$\Sigma = \{0,1\}$,

$\Gamma = \{0, \$\}$,

$F = \{q_1, q_4\}$, and

δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1			ε			Input
Stack:	0	\$	ε	0	\$	ε	0	\$	ε	Pop
q_1									$\{(q_2, \$)\}$	Push
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$ 2				4	5	
q_3			1	$\{(q_3, \varepsilon)\}$ 3				$\{(q_4, \varepsilon)\}$		
q_4										



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Input Pop Push

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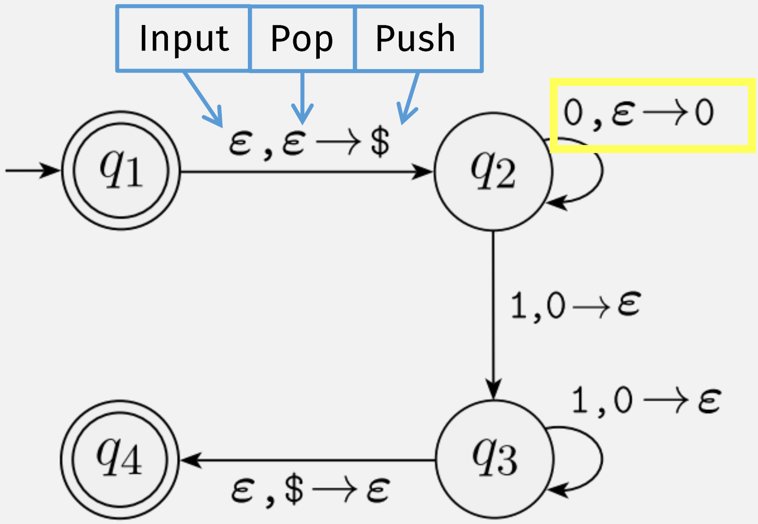
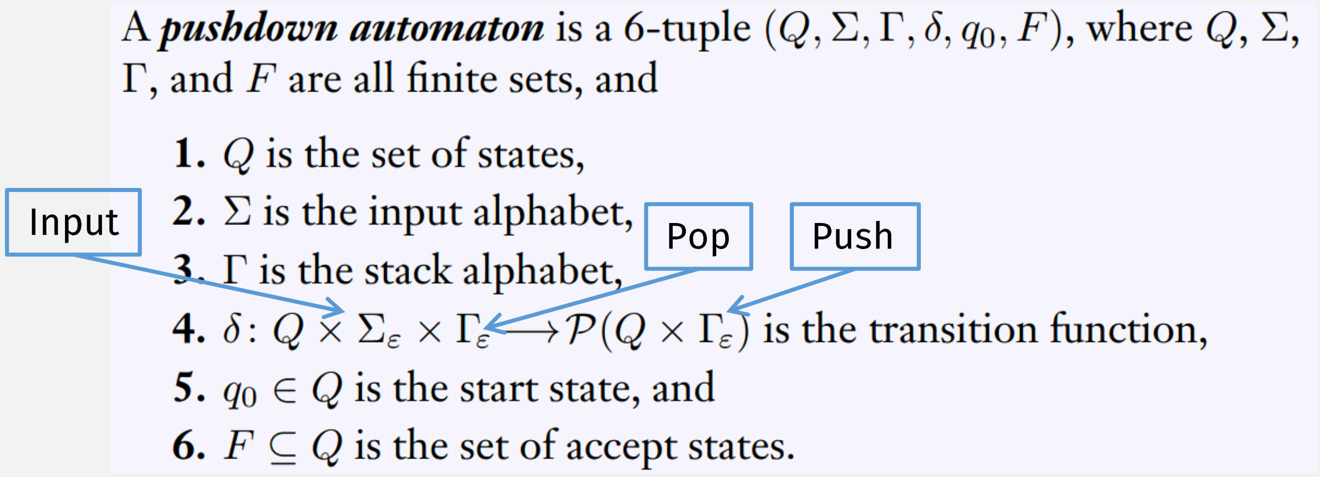
q_3				$\{(q_3, \varepsilon)\}$ 3				$\{(q_4, \varepsilon)\}$	
q_4									

Input

Pop

Push





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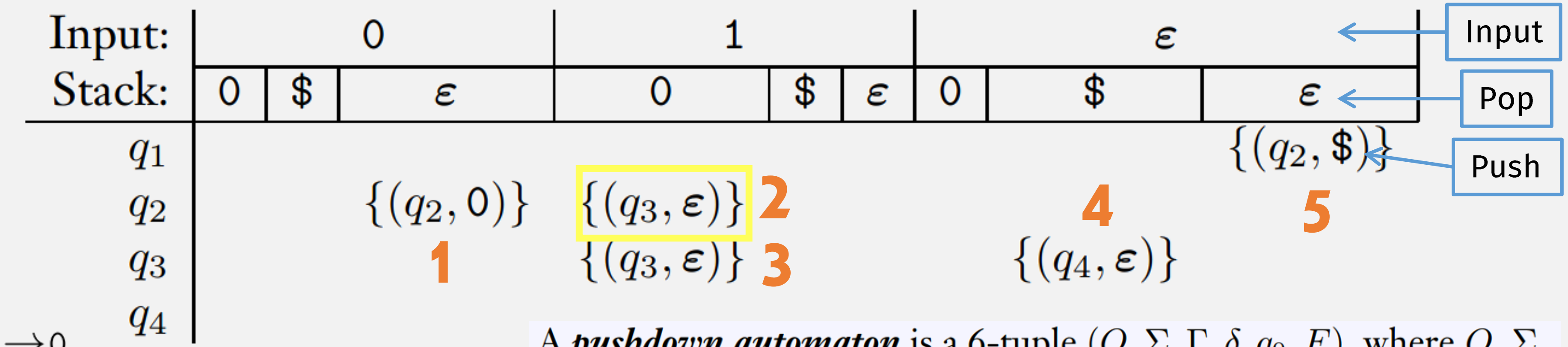

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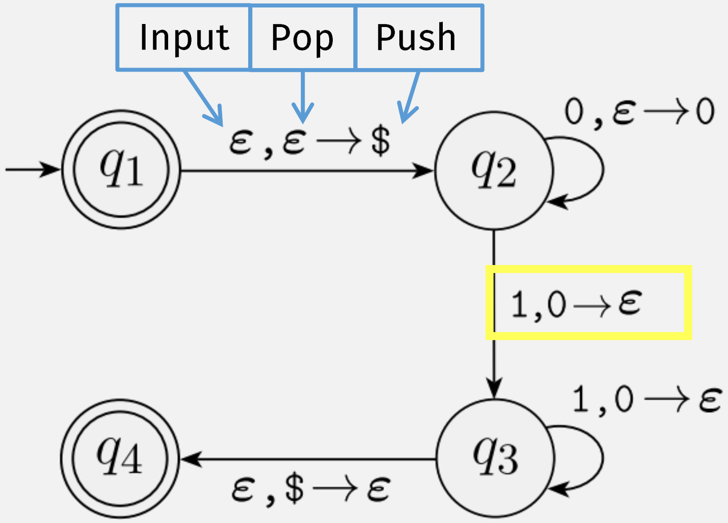

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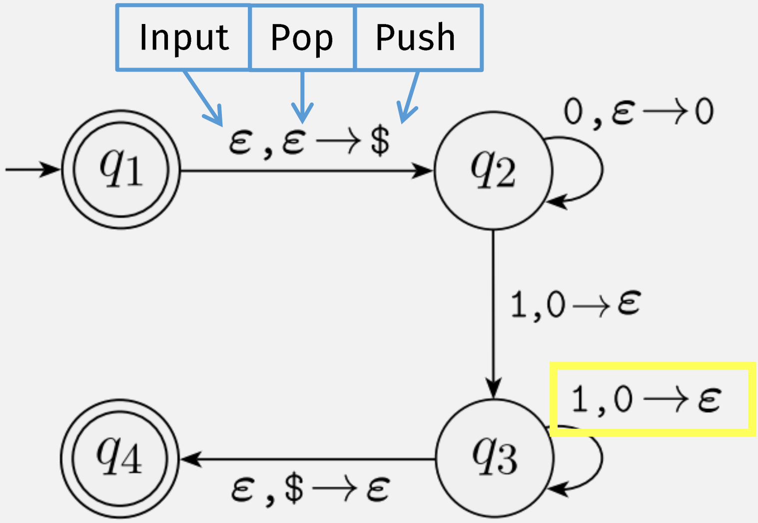
Input

Pop

Push



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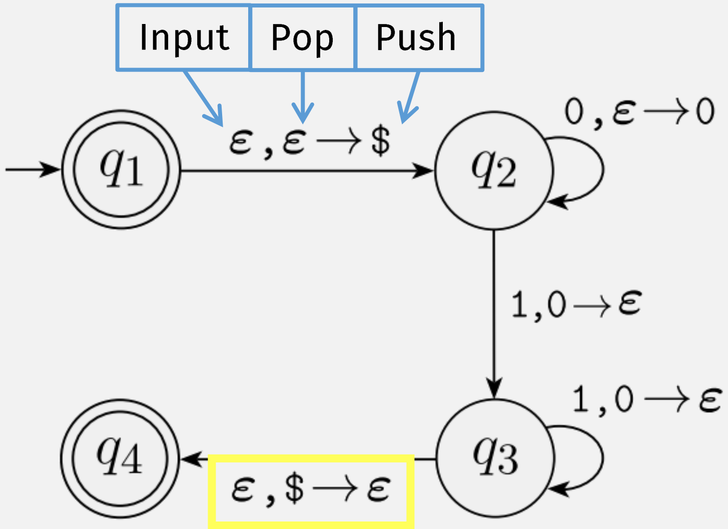

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Input

Pop

Push

In-class exercise: Fill in the blanks

$Q =$

$\Sigma =$

$\Gamma =$

$F =$

δ is given by the following table, wherein blank entries signify $\emptyset$.

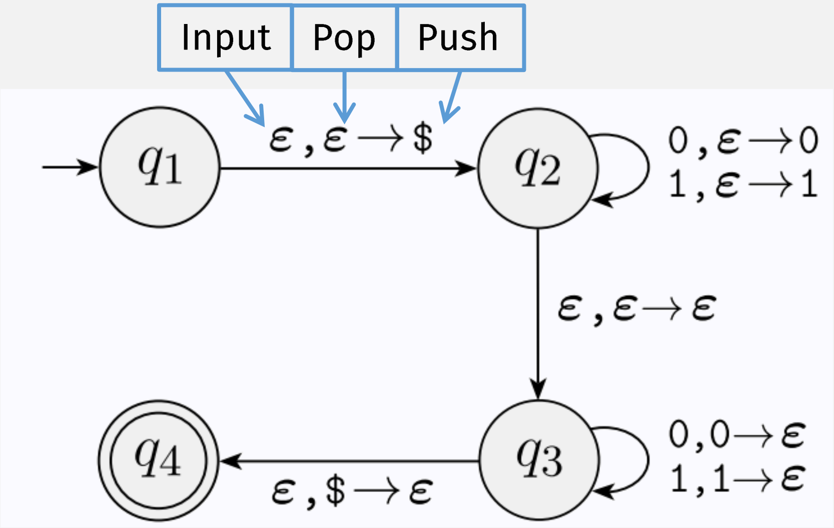
Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε

Input

Pop

State/ Push

PDA M_3 recognizing the language $\{ww^{\mathcal{R}} | w \in \{0,1\}^*\}$



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Input

Pop

Push

In-class exercise:
Fill in the blanks

$Q = \{q_1, q_2, q_3, q_4\}$,

$\Sigma = \{0,1\}$,

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$F = \{q_4\}$

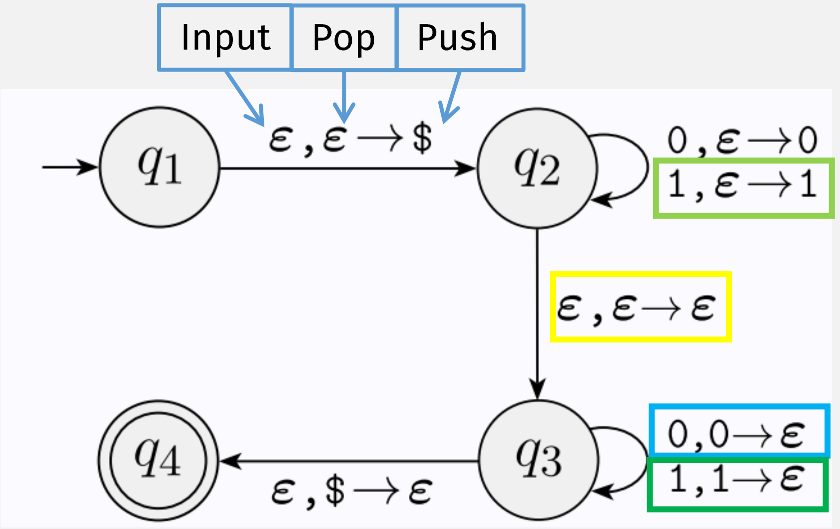
δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1				ε		
Stack:	0	\$	ε	0	1	\$	ε	0	\$	ε
q_1										$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$				$\{(q_2, 1)\}$			$\{(q_3, \varepsilon)\}$
q_3	$\{(q_3, \varepsilon)\}$				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4										

Input

Pop

State/Push



PDA M_3 recognizing the language $\{ww^{\mathcal{R}} | w \in \{0,1\}^*\}$

Flashback: DFA Computation Model

Informally

- "Program" = a finite automata
- Input = string of chars, e.g. "`1101`"

To run a "program":

- Start in "start state"
- Repeat:
  - Read 1 char;
  - Change state according to the transition table
- Result =
  - "**Accept**" if last state is "Accept" state
  - "**Reject**" otherwise

Formally (i.e., mathematically)

- $M = (Q, \Sigma, \delta, q_0, F)$
- $w = w_1 w_2 \cdots w_n$
- $r_0 = q_0$
- $r_i = \delta(r_{i-1}, w_i)$, for $i = 1, \ldots, n$
- M ***accepts*** w if

  sequence of states $r_0, r_1, \ldots, r_n$ in Q exists …

  with $r_n \in F$

A **sequence of states** represents a **DFA computation**

Flashback: A DFA Extended Transition Fn

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to Q$

- Domain:
  - Beginning state $q \in Q$ (not necessarily the start state)
  - Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range:
  - Ending state (not necessarily an accept state)

(Defined recursively)

This specifies the **sequence of states** for a **DFA** computation

- Base case: $\hat{\delta}(q, \varepsilon) = q$
- Recursive case: $\hat{\delta}(q, w) = \hat{\delta}(\delta(q, w_1), w_2 \cdots w_n)$

Last Time: PDA Configurations (IDs)

- A **configuration** (or **ID**) is a "snapshot" of a PDA's computation

- 3 components (q, w, γ) :

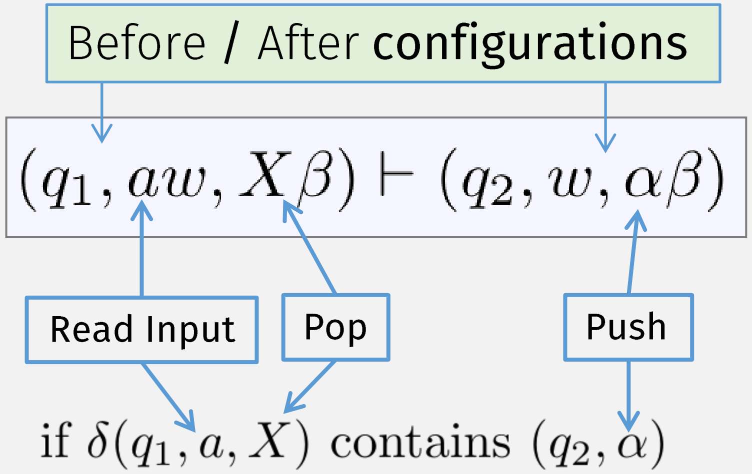
  q = the current state
  w = the remaining input string
  γ = the stack contents

A **sequence of configurations** represents a **PDA** computation

PDA Computation, Formally

$$P = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

Single-step



Before / After **configurations**

$$(q_1, aw, X\beta) \vdash (q_2, w, \alpha\beta)$$

Read Input

Pop

Push

$$\text{if } \delta(q_1, a, X) \text{ contains } (q_2, \alpha)$$

$$q_1, q_2 \in Q$$

$$a \in \Sigma \quad w \in \Sigma^*$$

$$X \in \Gamma \quad \beta, \alpha \in \Gamma^*$$

A **configuration** (q, w, γ) has three components

q = the current state

w = the remaining input string

γ = the stack contents

Extended

- Base Case

$$\boxed{I \overset{*}{\vdash} I \text{ for any ID } I}$$

- Recursive Case

$$\boxed{I \overset{*}{\vdash} J} \text{ if there exists some ID } K$$

$$\text{such that } I \vdash K \text{ and } K \overset{*}{\vdash} J$$

Single step

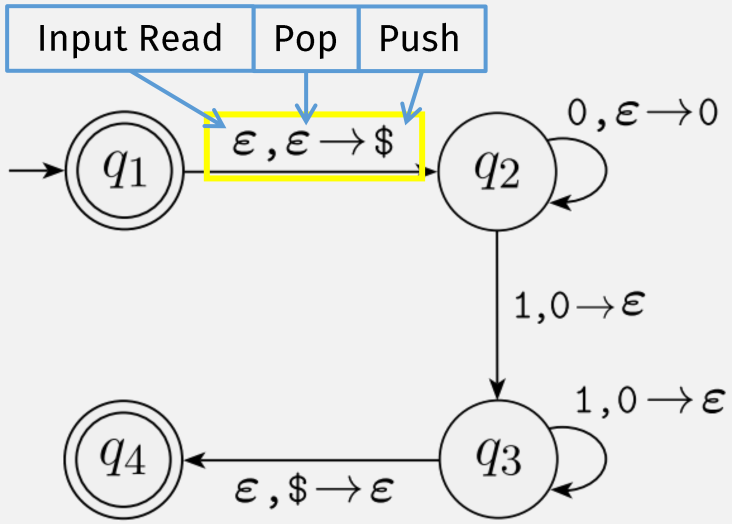
Recursive call

This specifies the **sequence of configurations** for a **PDA** computation

PDA Running Input String Example

State Remaining Input Stack

$$(q_1, 0011, \varepsilon)$$


Input Read Pop Push

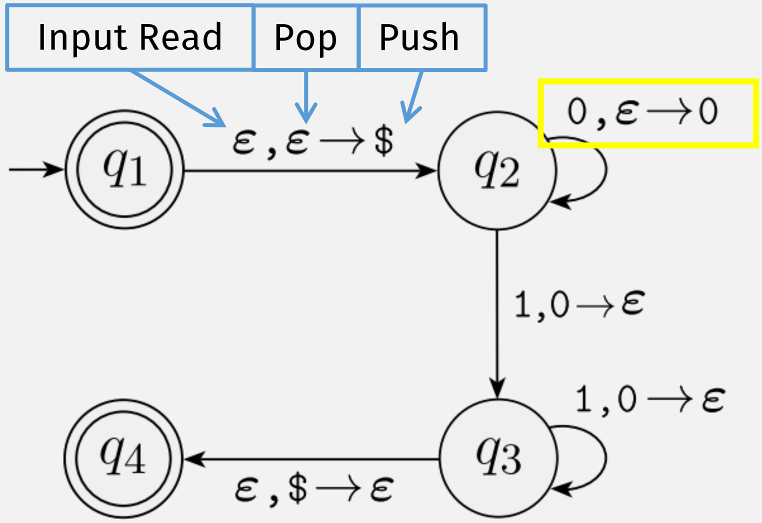
PDA Running Input String Example

State | Remaining Input | Stack

$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$

$$\vdash (q_2, 011, 0\$)$$

Read 0, push 0



PDA Running Input String Example

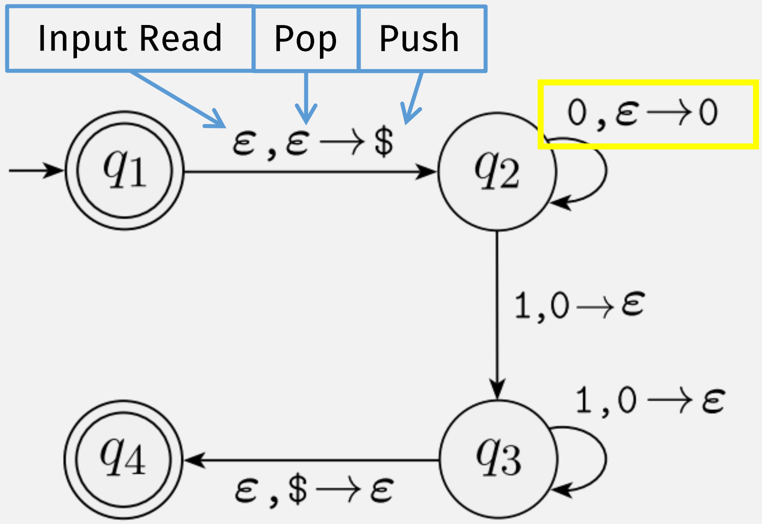
State

Remaining Input

Stack

$$
\begin{aligned}
(q_1, 0011, \varepsilon) &\vdash (q_2, 0011, \$) \\
&\vdash (q_2, 011, 0\$) \\
&\vdash (q_2, 11, 00\$)
\end{aligned}
$$

Read 0, push 0



PDA Running Input String Example

State

Remaining Input

Stack

$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$

$$\vdash (q_2, 011, 0\$)$$

$$\vdash (q_2, 11, 00\$)$$

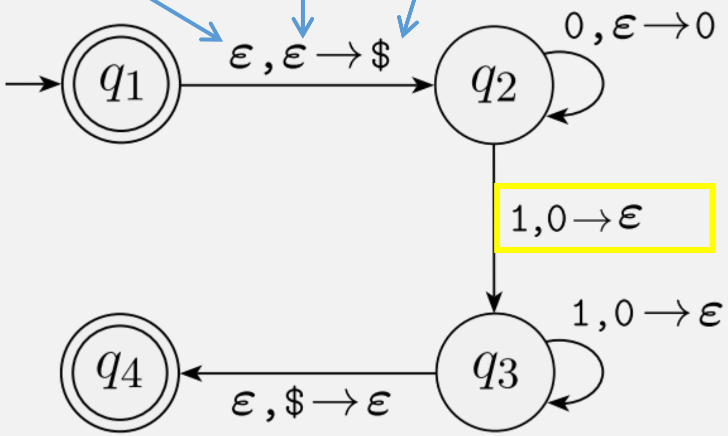
$$\boxed{\vdash (q_3, 1, 0\$)}$$

Read 1, pop 0

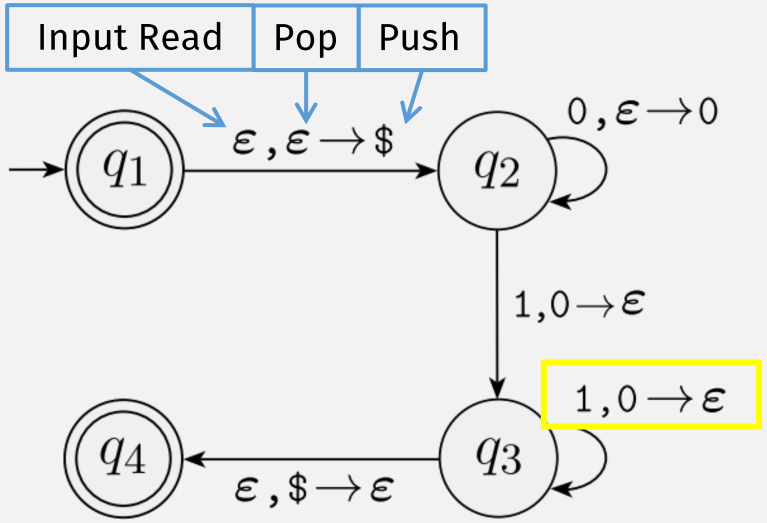
Input Read

Pop

Push



PDA Running Input String Example



State | Remaining Input | Stack

$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$
$$\vdash (q_2, 011, 0\$)$$
$$\vdash (q_2, 11, 00\$)$$
$$\vdash (q_3, 1, 0\$)$$
$$\boxed{\vdash (q_3, \varepsilon, \$)}$$

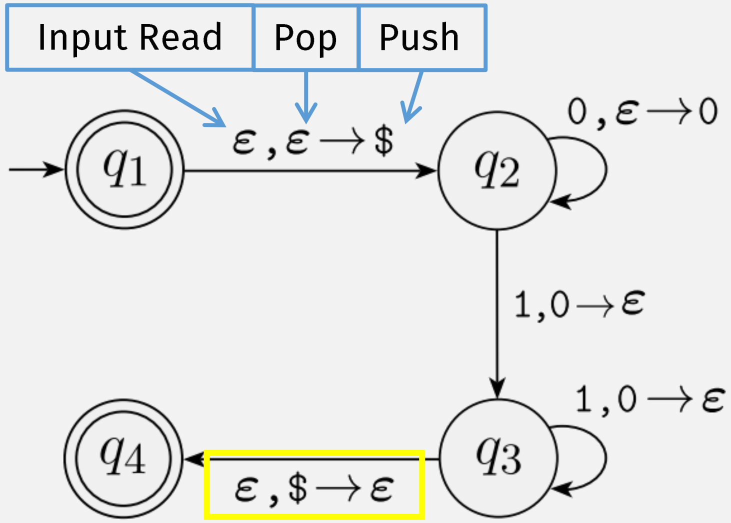
Read 1, pop 0

PDA Running Input String Example

State Remaining Input Stack

$$
\begin{aligned}
(q_1, 0011, \varepsilon) &\vdash (q_2, 0011, \$) \\
&\vdash (q_2, 011, 0\$) \\
&\vdash (q_2, 11, 00\$) \\
&\vdash (q_3, 1, 0\$) \\
&\vdash (q_3, \varepsilon, \$) \\
&\vdash \boxed{(q_4, \varepsilon, \varepsilon)}
\end{aligned}
$$

pop empty stack symbol

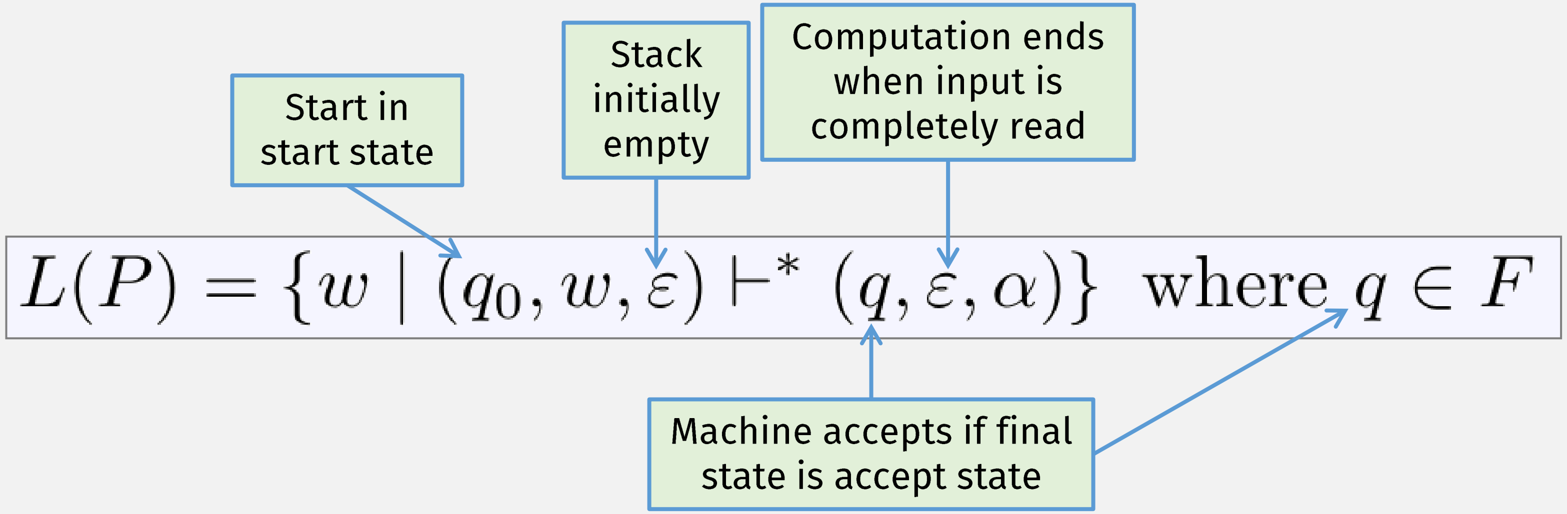


Flashback: Computation and Languages

- The **language** of a **machine** is the **set of all strings that it accepts**
- E.g., A **DFA** M **accepts** w **if** $\hat{\delta}(q_0, w) \in F$
- **Language** of $M = L(M) = \{ w \mid M$ **accepts** $w \}$

Language of a PDA

$$P = (Q, \Sigma, \Gamma, \delta, q_0, F)$$



Start in start state

Stack initially empty

Computation ends when input is completely read

$$L(P) = \{w \mid (q_0, w, \varepsilon) \vdash^* (q, \varepsilon, \alpha)\} \text{ where } q \in F$$

Machine accepts if final state is accept state

A **configuration** (q, w, γ) has three components

q = the current state

w = the remaining input string

γ = the stack contents

PDAs and CFLs?

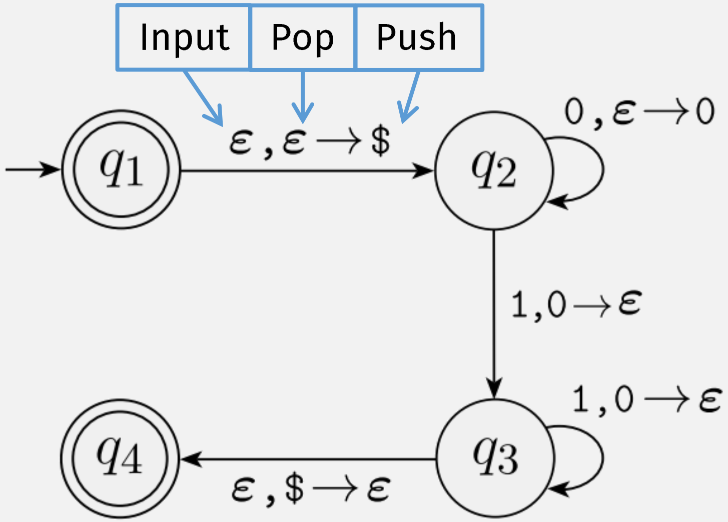
- **PDA** = NFA + a stack
  - Infinite memory
  - Can only read/write top location: Push/pop

- Want to prove: PDAs represent CFLs!

- We know: a CFL, by definition, is a language that is generated by a CFG

- Need to show: PDA ⇔ CFG

- Then, to prove that a language is a CFL, we can either:
  - Create a CFG, or
  - Create a PDA



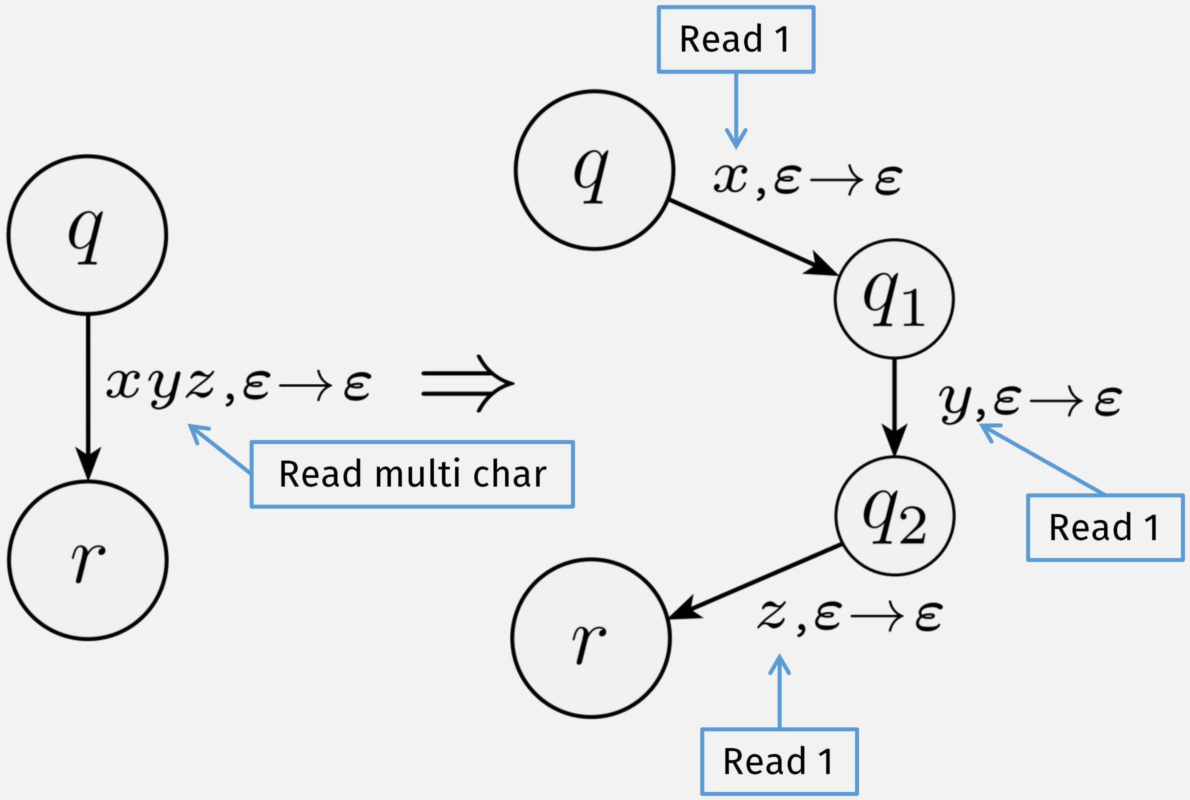
A lang is a CFL **iff** some PDA recognizes it

⇒ If **a language is a CFL**, then **a PDA recognizes it**

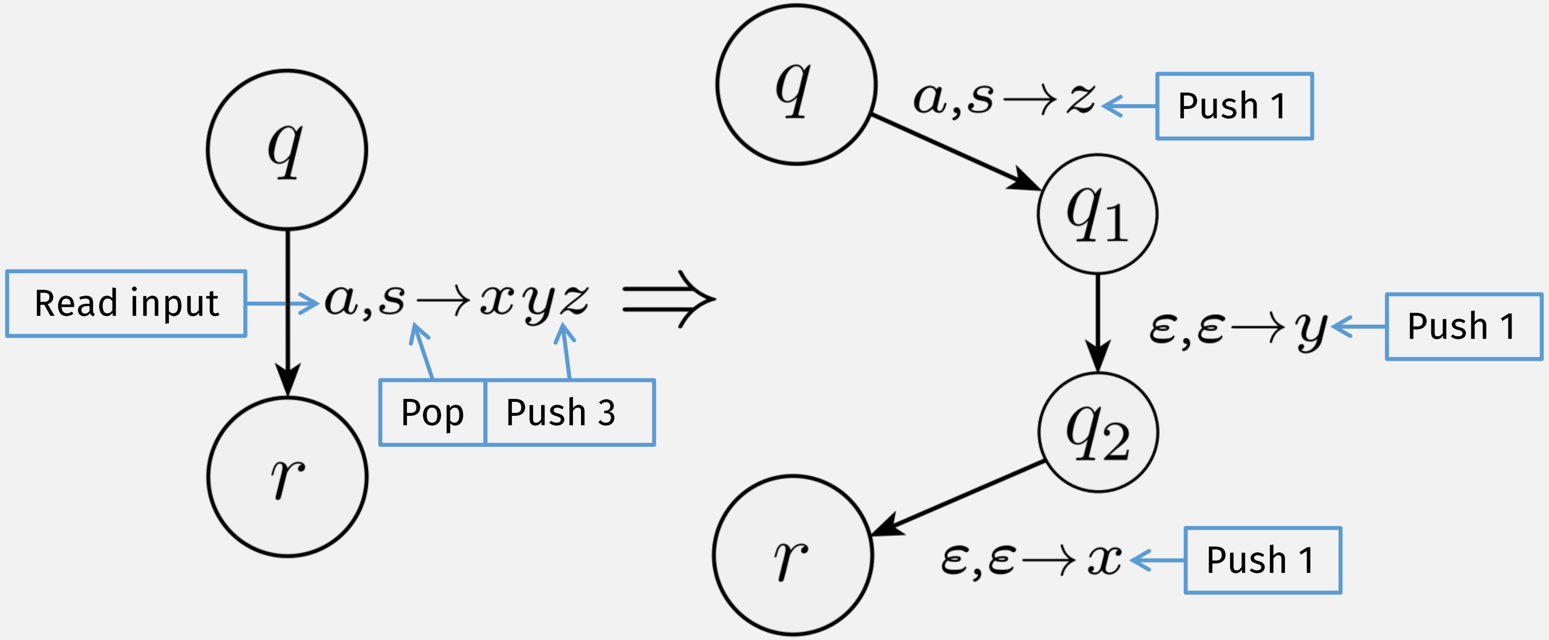
- We know: **A CFL has a CFG describing it** (definition of CFL)
- To prove this part: **show the CFG has an equivalent PDA**

⇐ If **a PDA recognizes a language,** then **it's a CFL**

Shorthand: Multi-Symbol Read Transition



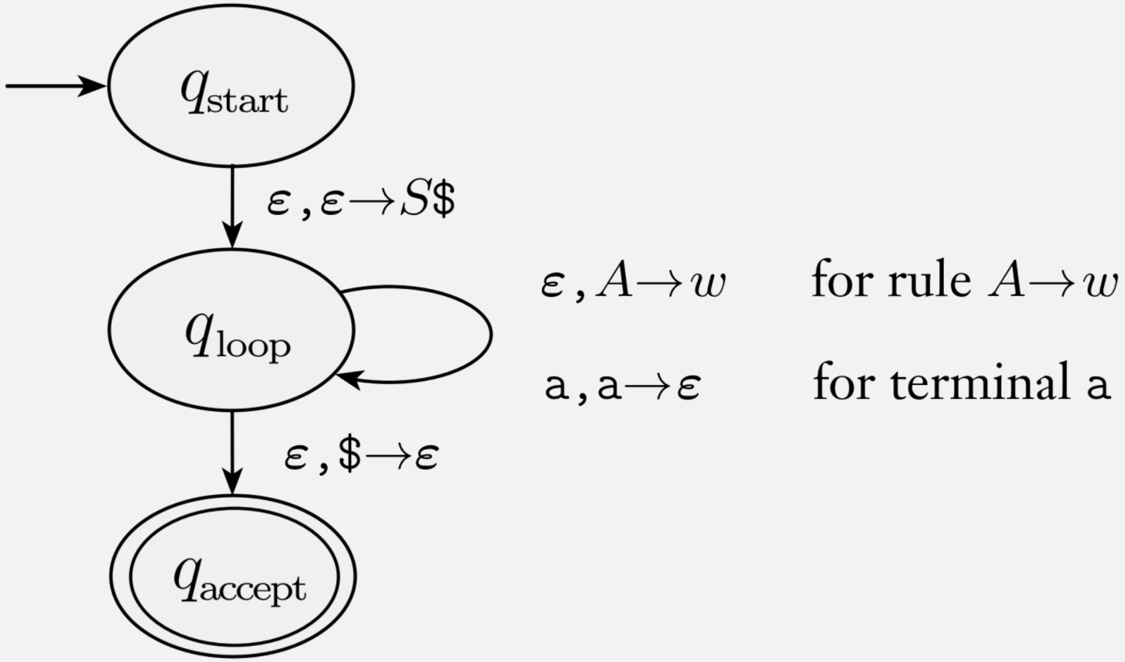
Shorthand: Multi-Stack Push Transition



Note the reverse order of pushes

CFG→PDA (sketch)

- Construct PDA from CFG such that:
  - PDA accepts input only if CFG generates it
- PDA:
  - simulates generating a string with CFG rules
  - by (nondeterministically) trying all rules to find the right ones



CFG→PDA (sketch)

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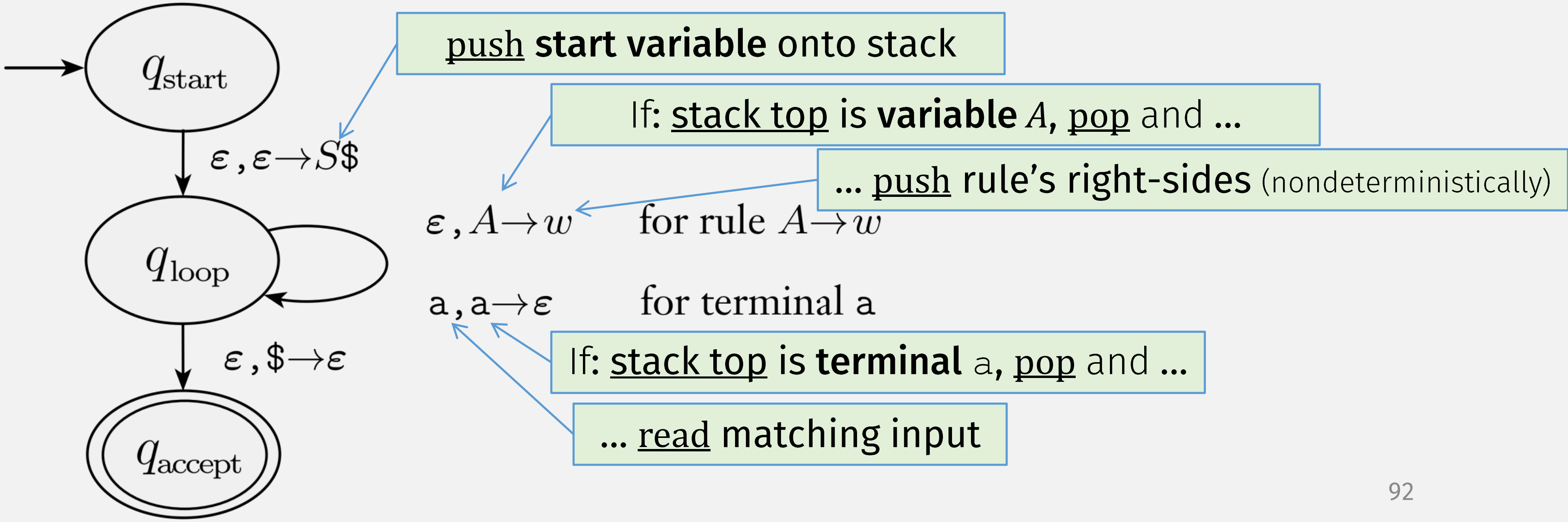


push **start variable** onto stack

If: stack top **is variable** A, pop and …

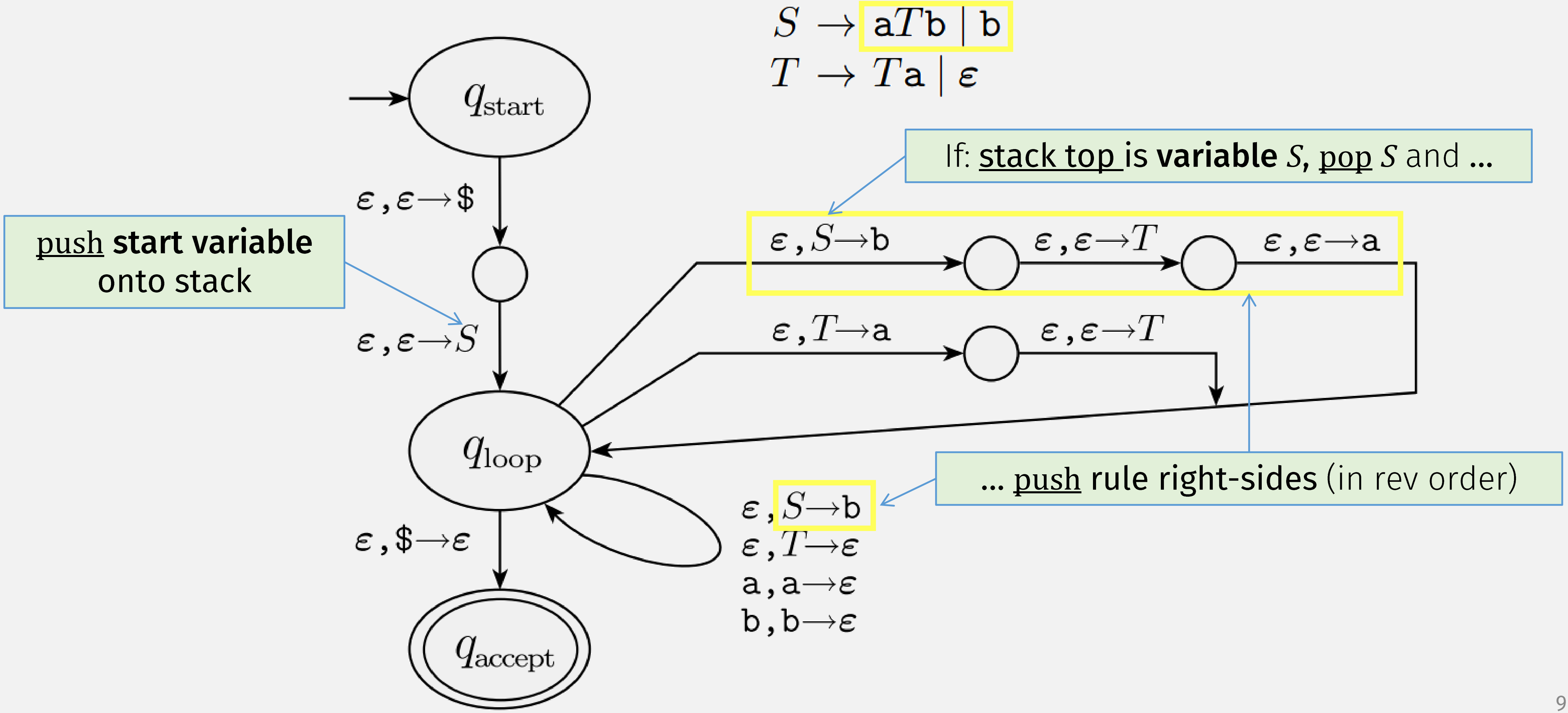
… push **rule's right-sides** (nondeterministically)

If: stack top **is terminal** a, pop and …

… read matching input

Example **CFG→PDA**

$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$
$T \to T\mathtt{a} \mid \varepsilon$



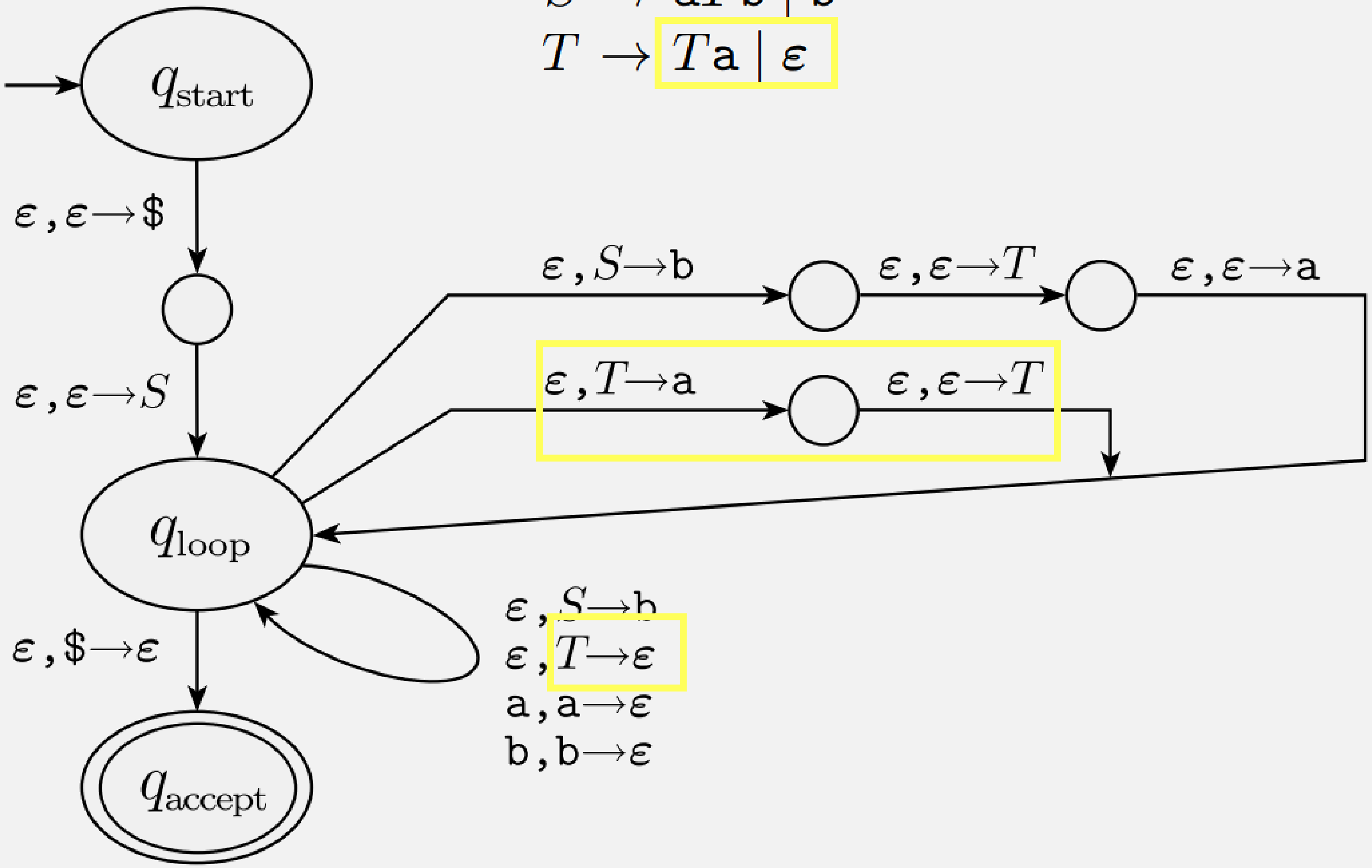
push **start variable** onto stack

If: stack top **is variable** *S*, pop *S* and …

… push **rule right-sides** (in rev order)

Example **CFG→PDA**

$$
\begin{aligned}
S &\to \mathtt{a}T\mathtt{b} \mid \mathtt{b} \\
T &\to T\mathtt{a} \mid \varepsilon
\end{aligned}
$$



Example **CFG→PDA**

$$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
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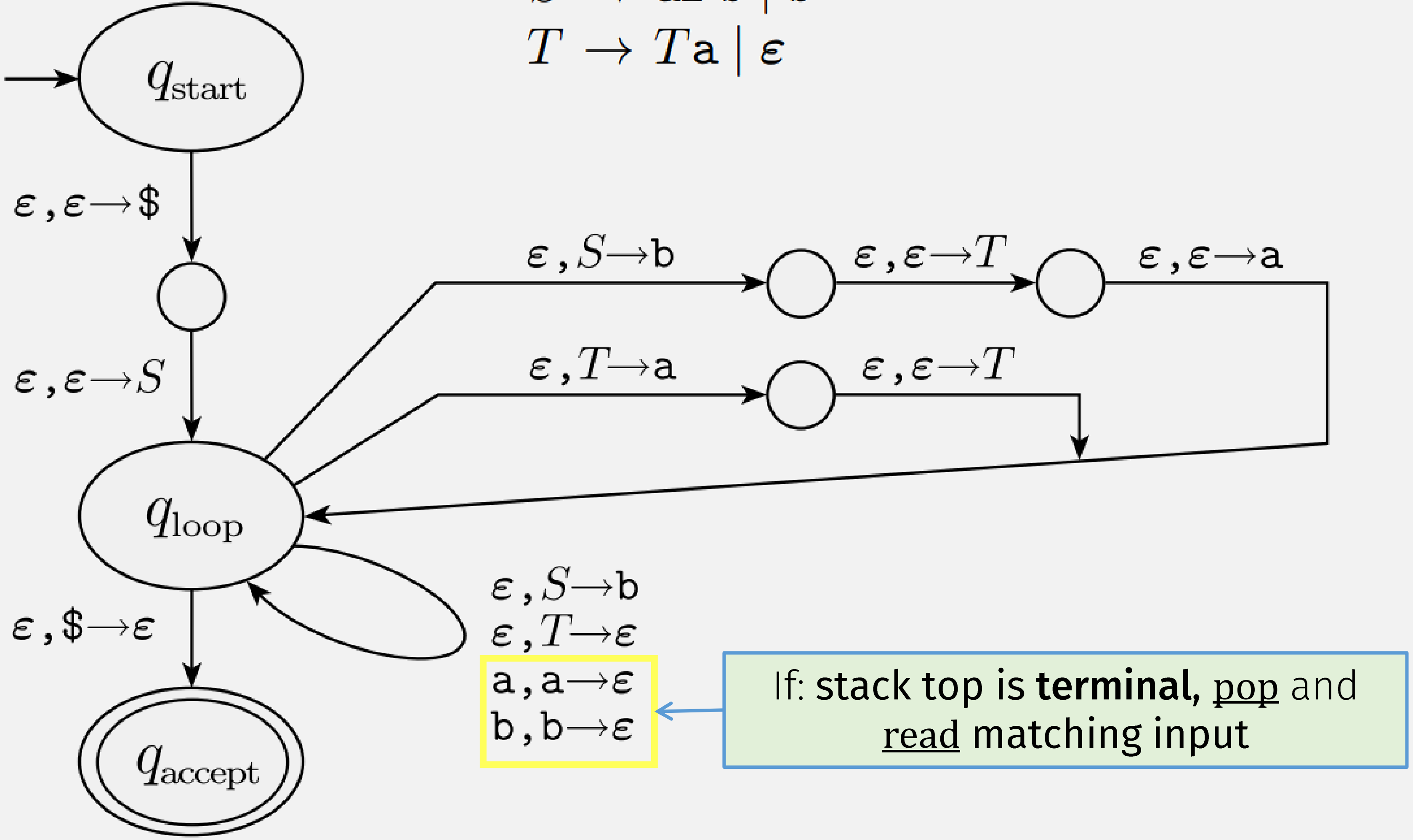


If: **stack top is terminal**, pop and read matching input

Example **CFG→PDA**

Example Derivation using CFG:

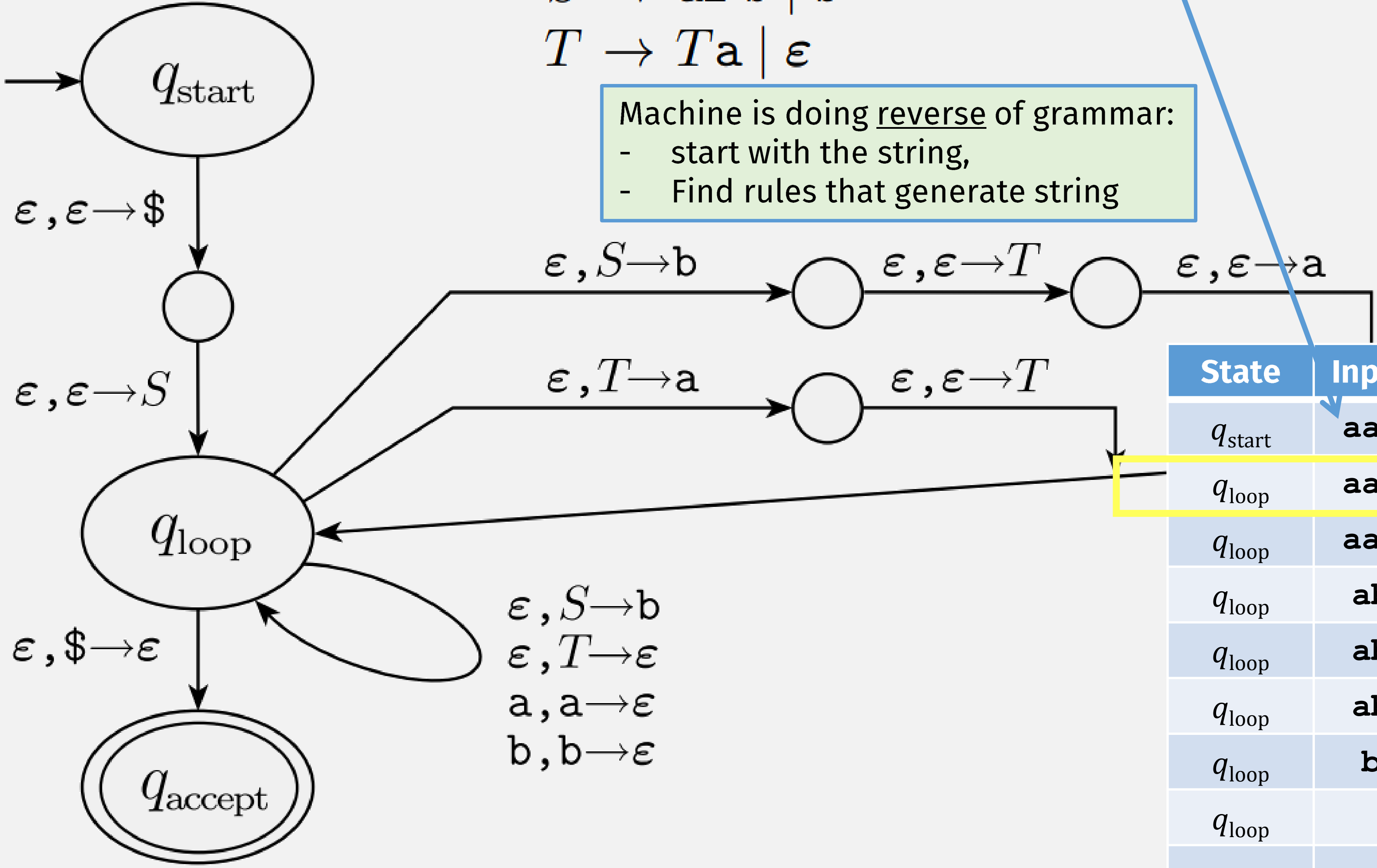
$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \rightarrow \mathtt{a}T\mathtt{b}$)

$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \rightarrow T\mathtt{a}$)

$\Rightarrow \mathtt{aab}$ (using rule $T \rightarrow \varepsilon$)

$$S \rightarrow \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \rightarrow T\mathtt{a} \mid \varepsilon$$

Machine is doing reverse of grammar:
- start with the string,
- Find rules that generate string



PDA Example

State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	$S\$$	
q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \rightarrow \mathtt{a}T\mathtt{b}$
q_{loop}	ab	$T\mathtt{b}\$$	
q_{loop}	ab	$T\mathtt{ab}\$$	$T \rightarrow T\mathtt{a}$
q_{loop}	ab	$\mathtt{ab}\$$	$T \rightarrow \varepsilon$
q_{loop}	b	$\mathtt{b}\$$	
q_{loop}		$\$$	
q_{accept}			

Example **CFG→PDA**

Example Derivation using CFG:

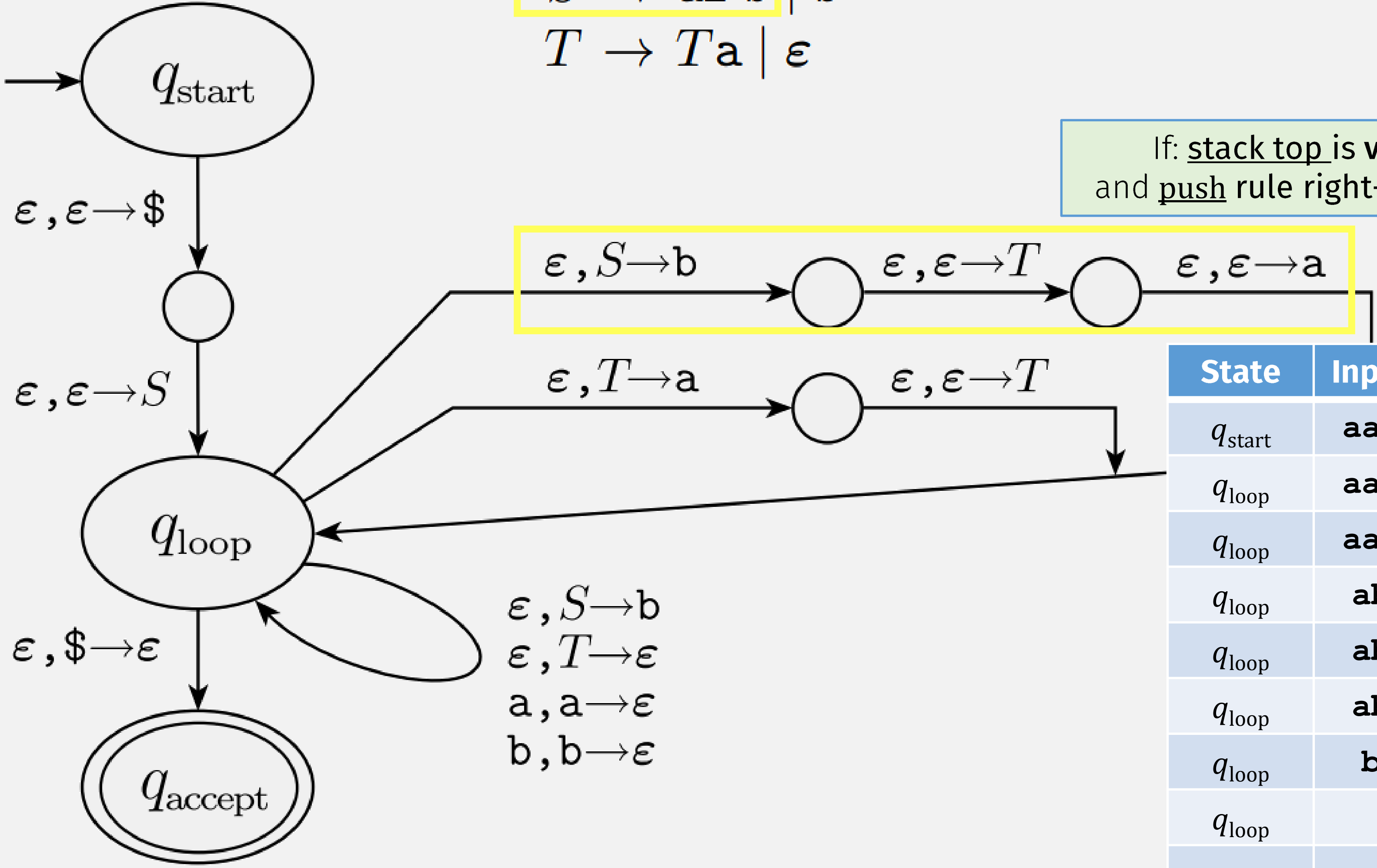
$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \rightarrow \mathtt{a}T\mathtt{b}$)

$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \rightarrow T\mathtt{a}$)

$\Rightarrow \mathtt{aab}$ (using rule $T \rightarrow \varepsilon$)

$$S \rightarrow \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \rightarrow T\mathtt{a} \mid \varepsilon$$

If: **stack top** is **variable** S, **pop** S and **push rule right-sides** (in rev order)



PDA Example

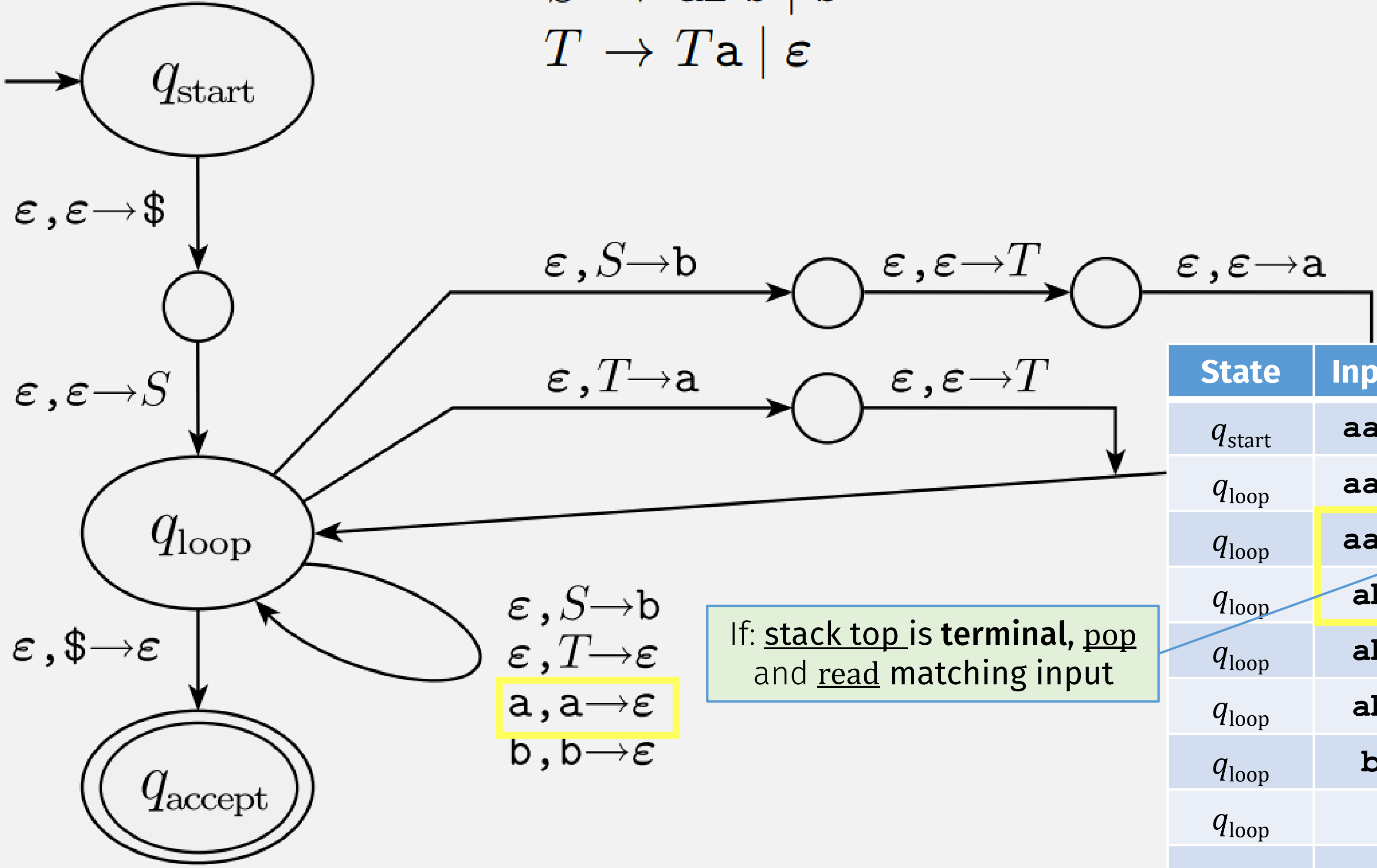
State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	$S\$$	
q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \rightarrow \mathtt{a}T\mathtt{b}$
q_{loop}	ab	$T\mathtt{b}\$$	
q_{loop}	ab	$T\mathtt{ab}\$$	$T \rightarrow T\mathtt{a}$
q_{loop}	ab	$\mathtt{ab}\$$	$T \rightarrow \varepsilon$
q_{loop}	b	$\mathtt{b}\$$	
q_{loop}		$\$$	
q_{accept}			

Example CFG→PDA

Example Derivation using CFG:

$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \to \mathtt{a}T\mathtt{b}$)
$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \to T\mathtt{a}$)
$\Rightarrow \mathtt{aab}$ (using rule $T \to \varepsilon$)

$$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \to T\mathtt{a} \mid \varepsilon$$



If: stack top is **terminal**, pop and read matching input

PDA Example

State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	$S\$$	
q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \to \mathtt{a}T\mathtt{b}$
q_{loop}	ab	$T\mathtt{b}\$$	
q_{loop}	ab	$T\mathtt{ab}\$$	$T \to T\mathtt{a}$
q_{loop}	ab	$\mathtt{ab}\$$	$T \to \varepsilon$
q_{loop}	b	$\mathtt{b}\$$	
q_{loop}		$\$$	
q_{accept}			

Example **CFG→PDA**

Example Derivation using CFG:
$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \to \mathtt{a}T\mathtt{b}$)
$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \to T\mathtt{a}$)
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$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$
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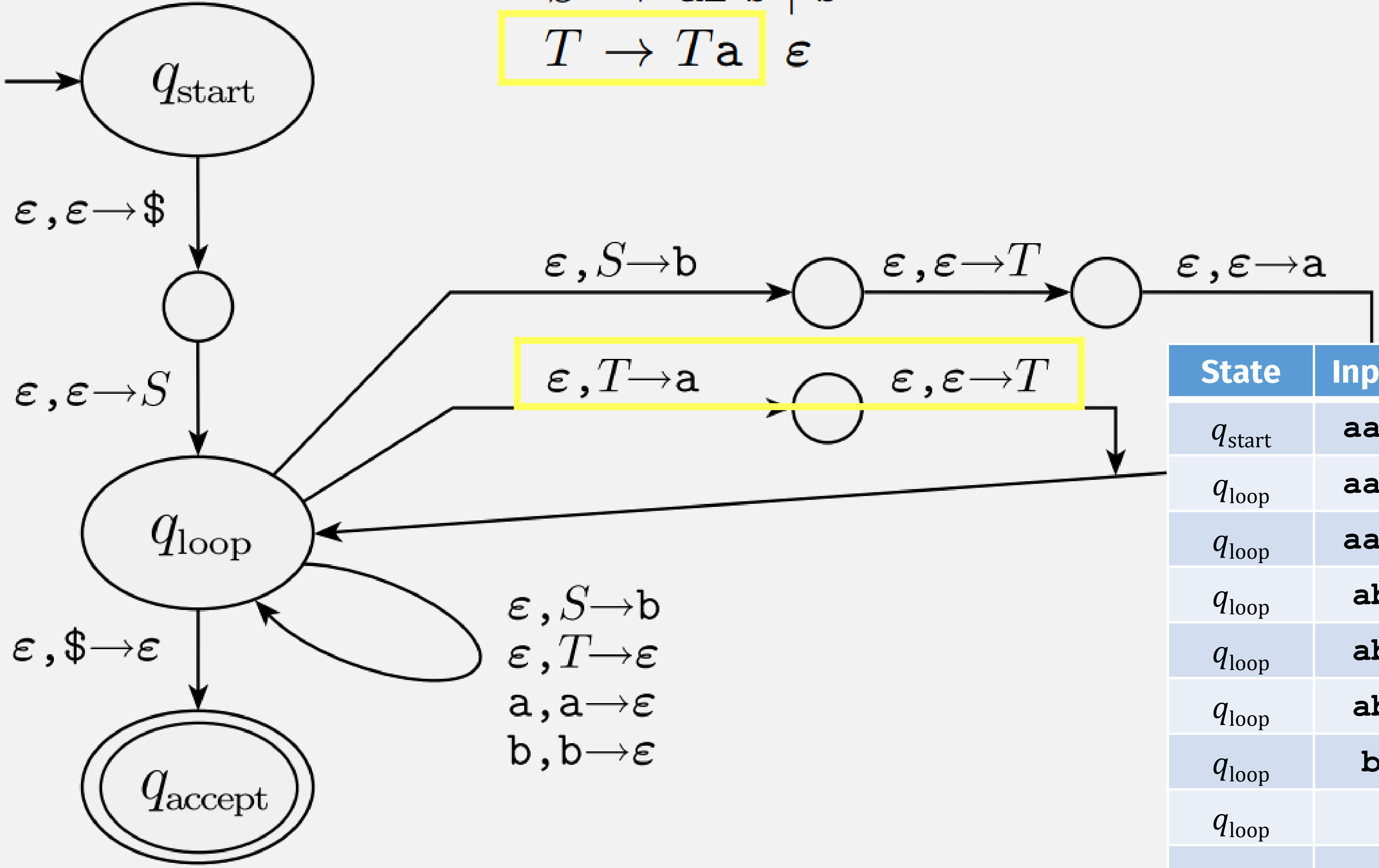

PDA Example

State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	$S\$$	
q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \to \mathtt{a}T\mathtt{b}$
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q_{loop}	ab	$T\mathtt{ab}\$$	$T \to T\mathtt{a}$
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q_{loop}		$\$$	
q_{accept}			

A lang is a CFL **iff** some PDA recognizes it

☑⇒ If **a language is a CFL,** then **a PDA recognizes it**

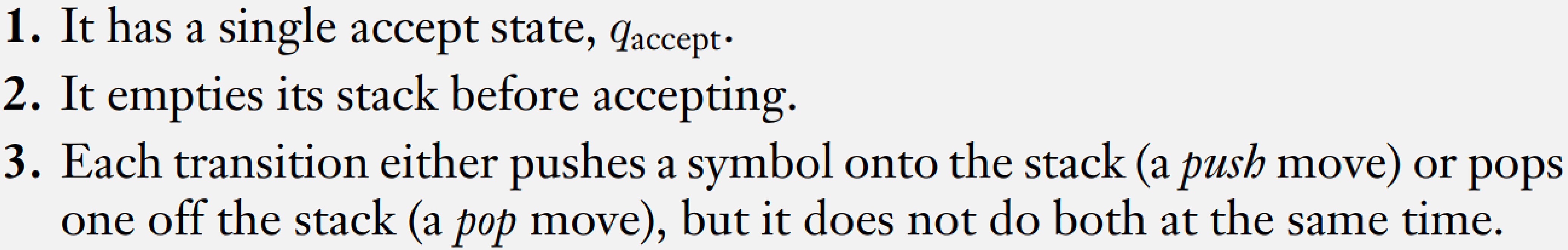
  - **Convert CFG→PDA**

⇐ If **a PDA recognizes a language,** then **it's a CFL**

  - To prove this part: show PDA has an equivalent CFG

PDA→CFG: Prelims

Before converting PDA to CFG, modify it so :



1. It has a single accept state, q_{accept}.
2. It empties its stack before accepting.
3. Each transition either pushes a symbol onto the stack (a *push* move) or pops one off the stack (a *pop* move), but it does not do both at the same time.

Important:
This doesn’t change the language recognized by the PDA

PDA P -> CFG G: Variables

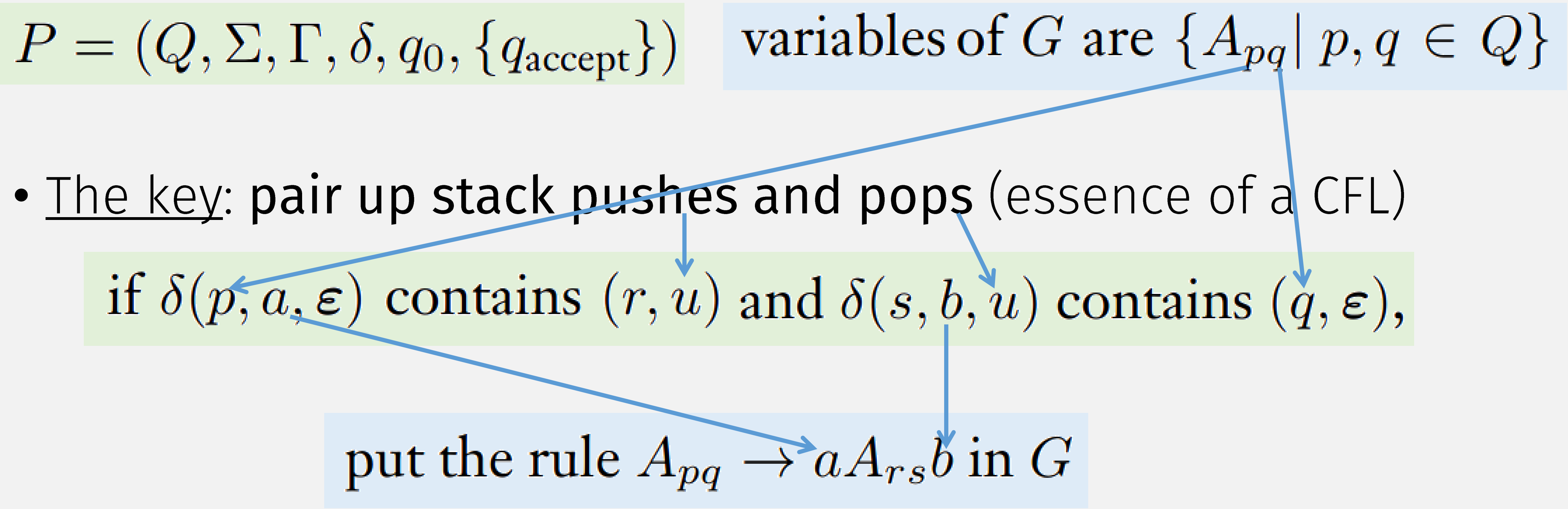
$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$ variables of G are $\{A_{pq} | \; p, q \in Q\}$

- Want: if P goes from state p to q reading input x, then some A_{pq} generates x
- So: For every pair of states p, q in P, add variable A_{pq} to G
- Then: connect the variables together by,
  - Add rules: $A_{pq} \rightarrow A_{pr}A_{rq}$, for each state r
  - These rules allow grammar to simulate every possible transition
  - (We haven't added input read/generated terminals yet)

The Key IDEA

- To add terminals: **pair up stack pushes and pops** (essence of a CFL)

PDA P -> CFG G: Generating Strings



$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$

variables of G are $\{A_{pq} \mid p, q \in Q\}$

- <u>The key</u>: **pair up stack pushes and pops** (essence of a CFL)

if $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),

put the rule $A_{pq} \rightarrow aA_{rs}b$ in G

PDA $\boldsymbol{P}$ -> CFG $\boldsymbol{G}$: Generating Strings

$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$

variables of G are $\{A_{pq} | \, p, q \in Q\}$

- <u>The key</u>: **pair up stack pushes and pops** (essence of a CFL)

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PDA P -> CFG G: Generating Strings

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- <u>The key</u>: **pair up stack pushes and pops** (essence of a CFL)

if $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),

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A language is a CFL ⇔ A PDA recognizes it

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- Convert **CFG→PDA**

☑⇐ If **a PDA recognizes a language,** then **it's a CFL**

- Convert **PDA→CFG**

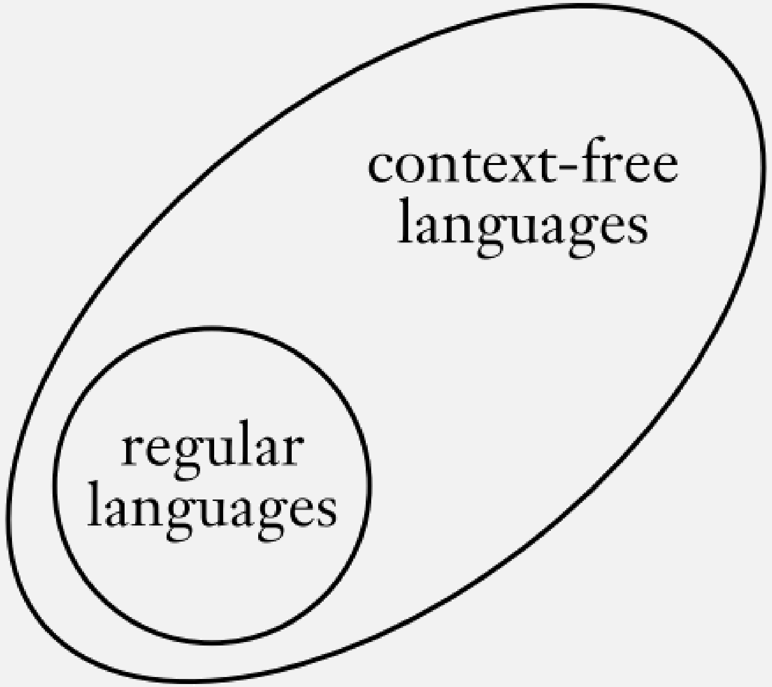
∎

Regular Languages are CFLs: 3 Proofs

- DFA → CFG
  - HW?

- NFA → CFG
  - NFA → PDA (with no stack moves) → CFG
  - Just now

- Regular expression → CFG
  - HW?



Check-in Quiz 3/8

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