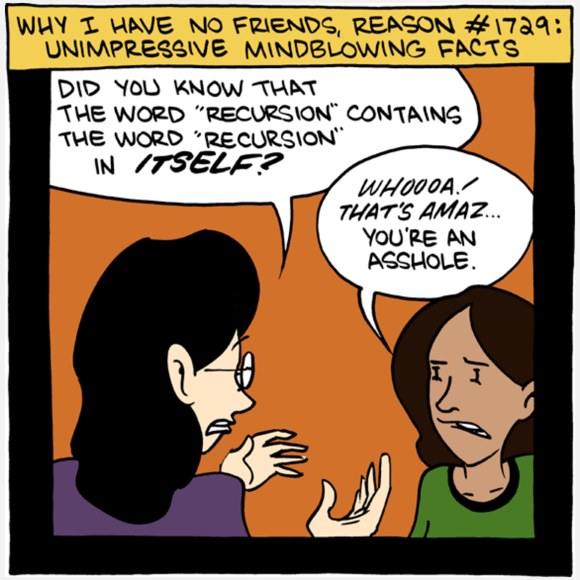
UMass Boston Computer Science

CS450 High Level Languages (section 2)

Recursion in the Lambda Calculus

Monday, October 23, 2023



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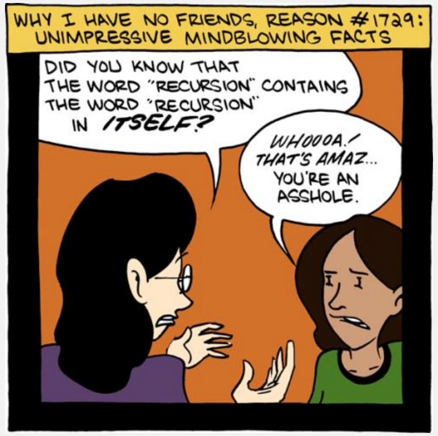
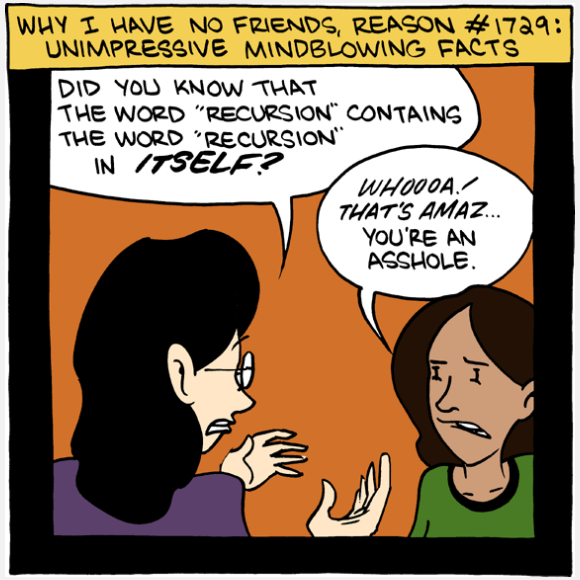


Logistics

- HW 4 in
  - ~~due: Sun 10/22 11:59 pm EST~~

- HW 5 out
  - due: Sun 10/29 11:59 pm EST



From Lecture 1

NOTE: This hierarchy is *approximate*

"high" level
(easier for humans to understand)

"Computation" = "arithmetic" of expressions

"declarative"

Core model: **Lambda Calculus**

"Computation" = sequence of instructions / statements

"imperative"

Core model: **Turing Machines**

"low" level
(runs on cpu)

English	
Specification langs	Types? pre/post cond?
Markup (html, markdown)	tags
Database (SQL)	queries
Logic Program (Prolog)	relations
Lazy lang (Haskell, R)	Delayed computation
Functional lang (Racket)	Expressions (no stmts)
JavaScript, Python	"eval"
C# / Java	GC (no alloc, ptrs)
C++	Classes, objects
C	Scoped vars, fns
Assembly Language	Named instructions
Machine code	0s and 1s

This class: **how to program in a high-level more "human friendly" way**

"Nicer" for humans to use

Last Time

The Lambda (λ) Calculus

- A **"programming language"** consisting of only:
  - Lambda functions
  - Function application

- **Equivalent** in **"computational power"** to
  - Turing Machines
  - Your favorite programming language!

Last Time

Church Numerals

```
;; A ChurchNum is a function with two arguments:
;; "f" : a function to apply
;; "base" : a base ("zero") value to apply to
;;
;; For a specific number, its "Church" representation
;; applies the given function that number of times
```

```
(define czero
  (lambda (f base) base))
```

f applied zero times

```
(define cone
  (lambda (f base) (f base)))
```

f applied one time

```
(define ctwo
  (lambda (f base) (f (f base))))
```

f applied two times

```
(define cthree
  (lambda (f base) (f (f (f base)))))
```

f applied three times

Church "Add1"

```
;; cplus1 : ChurchNum -> ChurchNum
;; "Adds" 1 to the given Church num
```

```
(define cplus1
  (lambda (n)
    (lambda (f base)
      (f (n f base)))))
```

Input ChurchNum

Returns ChurchNum that …

(we know "n" will apply f n times)

… adds an extra application of f

```
(define czero
  (lambda (f base) base))
```

```
(define cone
  (lambda (f base) (f base)))
```

```
(define ctwo
  (lambda (f base) (f (f base))))
```

```
(define cthree
  (lambda (f base) (f (f (f base)))))
```

Church Addition

```
;; cplus : ChurchNum ChurchNum -> ChurchNum
;; “Adds” the given ChurchNums together
```

```
(define cplus
  (lambda (m n)
    (lambda (f base)
      (m f (n f base)))))
```

Input ChurchNums

Returns a ChurchNum that ...

(we know “n” will apply f n times)

... adds “m” extra applications of f

```
(define czero
  (lambda (f base) base))
```

```
(define cone
  (lambda (f base) (f base)))
```

```
(define ctwo
  (lambda (f base) (f (f base))))
```

```
(define cthree
  (lambda (f base) (f (f (f base)))))
```

Last Time

Church Booleans

```
;; A ChurchBool is a function with two arguments,
;; where the representation of:
;; “true” returns the first arg, and
;; “false” returns the second arg
```

```
(define ctrue
  (lambda (a b) a))
```

Returns first arg

```
(define cfalse
  (lambda (a b) b))
```

Returns second arg

Review: "And"

The truth table of $A \wedge B$:

A	B	$A \wedge B$
True	True	True
True	False	False
False	True	False
False	False	False

When A = True,
then $\text{And}(A, B) = B$

When A = False,
then $\text{And}(A, B) = A$

Church "And"

```
;; cand: ChurchBool ChurchBool-> ChurchBool
;; "ands" the given ChurchBools together
```

The truth table of $A \wedge B$:

A	B	$A \wedge B$
True	True	True
True	False	False
False	True	False
False	False	False

When A = True, want And(A, B) = B ☑

When A = False, want And(A, B) = A ☑

```
(define cand
  (lambda (A B)
    (A B A)))
```

```
(define ctrue
  (lambda (a b) a))
```

(Returns first arg)

```
;; if A = ctrue
;; then (A B A) = B
;; want (cand A B) = B
```

☑

```
(define cfalse
  (lambda (a b) b))
```

(Returns second arg)

```
;; if A = cfalse
;; then (A B A) = A
;; want (cand A B) = A
```

☑

Last Time

Church Pairs (Lists)

```
;; A ChurchPair<X,Y> 1-arg function, where
;; the arg fn is applied to (i.e., "selects") the X and Y data values
```

```
;; ccons: X Y -> ChurchPair<X,Y>
(define ccons
  (lambda (x y)
    (lambda (get)
      (get x y))))
```

```
(define cfirst
  (lambda (cc)
    (cc (lambda (x y) x))))
```

“Gets” the first item

```
(define csecond
  (lambda (cc)
    (cc (lambda (x y) y))))
```

“Gets” the second item

Last Time

The Lambda (λ) Calculus

- A **“programming language”** consisting of only:
  - Lambda functions
  - Function application

- “Language” has:
  - Numbers
  - Booleans and conditionals
  - Lists
  - …
  - Recursion?

Recursion in the Lambda Calculus

Q: How can we write recursive programs with no-name lambdas?

Q: Is there a way for a lambda program to reference itself?

From Lecture 2

Lambda Program that Knows "Itself"

- **Program that runs "itself" repeatedly** (i.e., **it infinite loops**):

Function (takes one argument)

```
((λ (x) (x x))
  (λ (x) (x x)))
```

Function applies argument (function) **to itself**

Argument (is also function)

Result is:	**The same program** (i.e., the **program "itself"**)

- **Can we do something else besides loop?**

Lambda Program that Prints "Itself"

- Program that prints "itself":

Function (takes one argument)

```
((λ (x) (print2x x))
 "(λ (x) (print2x x))")
```

Apply function `print2x` to string argument

Argument (string)

Result is: **The same program** (i.e., the **program** "itself")

```
(define (print2x str)
  (printf "(~a\n ~v)\n" str str)))
```

Line break

(could have inlined this)

Function

Argument

Lambda Program that Prints "Itself"

- Program that prints "itself":

Also "itself" (part of program)

```
((λ (x) (print2x x))
 "(λ (x) (print2x x))")
```

"Itself"
(whole program)

- Q: Which part of the program is "itself"?

Lambda Program that Knows “Itself”

- Program that runs “itself” repeatedly (i.e., it infinite loops):

“the recursive call”

Also “itself” (part of program)

```
((λ (x) (x x))
  (λ (x) (x x)))
```

“Itself”
(whole program)

Insight:
“itself” = “the recursive call”

- Q: Which part of the program is “itself”?
- Can we do something more useful with “the recursive call”?

Delay “the recursive call”

What do we do with this?

Delayed “recursive call”

“the recursive call”

“the recursive call”

```
((λ (x) (x x))
 (λ (x) (x x)))
```

```
((λ (x) (λ (v) ((x x) v)))
 (λ (x) (λ (v) ((x x) v))))
```

Add a function parameter

Give “the recursive call” to another function that needs it

```
(λ (f)
  ((λ (x) (f (λ (v) ((x x) v))))
   (λ (x) (f (λ (v) ((x x) v))))))
```

What function “needs” a recursive call?
A Recursive function!

A Recursive Function

```
(define (factorial n)
  (if (zero? n)
   1
   (* n (factorial (sub1 n)))))
```

A Recursive Function, as lambda

```
(define factorial
  (λ (n)
    (if (zero? n)
     1
     (* n (factorial (sub1 n))))))
```

A Recursive Function without recursion

```
(define factorial
  (λ (n)
    (if (zero? n)
     1
     (* n (THE-RECURSIVE-CALL (sub1 n))))))
```

Where does this come from?

Make it a parameter!

A Recursive Function without recursion

```
(define factorial
  (λ (THE-RECURSIVE-CALL)
    (λ (n)
      (if (zero? n)
       1
       (* n (THE-RECURSIVE-CALL (sub1 n)))))))
```

Make “the recursive call” a parameter

A Recursive Function without recursion

```
(define factorial factorial-maker
  (λ (THE-RECURSIVE-CALL)
    (λ (n)
      (if (zero? n)
       1
       (* n (THE-RECURSIVE-CALL (sub1 n)))))))
```

Make “the recursive call” a parameter

Delay “the recursive call”

“the recursive call”

```
((λ (x) (x x))
 (λ (x) (x x)))
```

Delayed
“recursive call”

“the recursive call”

```
((λ (x) (λ (v) ((x x) v)))
 (λ (x) (λ (v) ((x x) v))))
```

Function that needs
“recursive call”

```
(λ (f)
  ((λ (x) (f (λ (v) ((x x) v))))
   (λ (x) (f (λ (v) ((x x) v))))))
```

f could be “fact-maker”

Y Combinator

"the recursive call"

```
((λ (x) (x x))
 (λ (x) (x x)))
```

Delayed "recursive call"

"the recursive call"

```
((λ (x) (λ (v) ((x x) v)))
 (λ (x) (λ (v) ((x x) v))))
```

Y Combinator "creates" recursive functions

f could be "`fact-maker`"

```
(λ (f)
  ((λ (x) (f (λ (v) ((x x) v))))
   (λ (x) (f (λ (v) ((x x) v))))))
```

Code Demo

Check-In Quiz 10/23

on gradescope

(due 1 minute before midnight)